Final TMath 402 Winter 2026

Recall you may use a two-sided 8.5" by 11" sheet of notes (or surface with equal area) with what ever you like written or typed on it. Additionally, a non-internet accessing calculator, or Desmos Test Mode can be used.

Show your work for the following problems to earn full marks.

1. [3] Prove or provide a counterexample for the following statement: Let S_{10} be the permutation group on ten elements. For all $\sigma \in S_{10}$, $\sigma^{10} = ()$, where $()$ is the identity permutation.

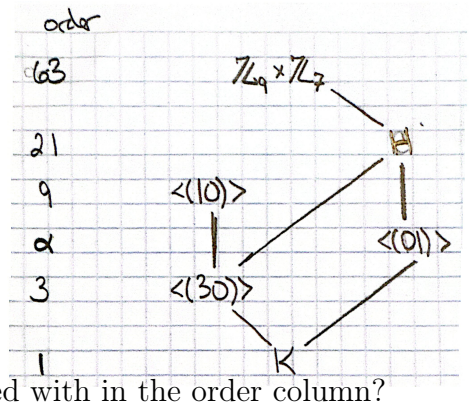
2. [3] Prove or provide a counterexample for the following statement: We can define a ring structure on the set $R = \{a + b\sqrt{5} | a, b \in \mathbb{Z}\}$.

3. Consider $\theta : D_4 \to \mathbb{Z}_4$ defined by $\theta(r^k f) = k$.

   (a) [2] If θ is a homomorphism, find $\theta(f)$. Show your computations/justification.

   (b) [2] Can θ be extended to define a homormophism? Justify your answer.

4. Consider the partial subgroup lattice for $\mathbb{Z}_9 \times \mathbb{Z}_7$ shown to the right.



   (a) [1] Find a generating set for the subgroup denoted with K.

   (b) [2] Find a generating set for the subgroup denoted with H.

   (c) [1] What number should α be replaced with in the order column?

   (d) [1] One edge is missing. Add the missing edge to the subgroup lattice.

   (e) [2] Could $\mathbb{Z}_9 \times \mathbb{Z}_7$ form a field with component-wise addition and multiplication reduced by the appropriate modulo ? Why or why not?

5. Consider a homomorphism $\phi : \mathbb{Z}_{30} \to \mathbb{Z}_{45}$.

   (a) [3] What are the possible sizes for the image of ϕ? Clearly provide justification and reasoning.

   (b) [4] How many different homomorphisms can be defined for ϕ with $\ker(\phi) = \{0, 15\}$.

6. Let $X = \{1, 2, 3, 4, 5\}$ and let S be the set of all permutations on X.

   (a) [5] Let $\alpha = (1, 2, 4)$ and $\beta = (1, 5)$. Find the following:

      i. α as a product of transpositions

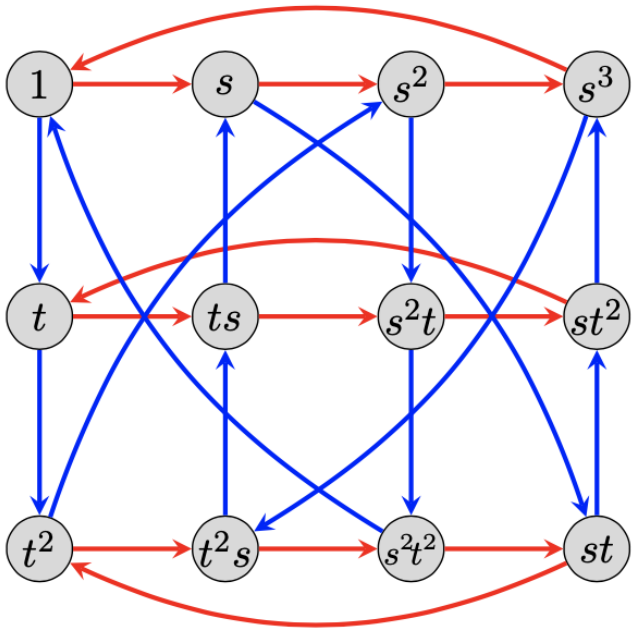
      ii. α^{-1}

      iii. $\alpha \circ \beta$

      iv. Does α and β commute? Justify your answer.

   (b) [4] Define a subset $H = \{\sigma \in S | \sigma(3) = 3\}$. That is, all permutations that do not move 3 (or stabilize 3). For example, α from part (a) is in H since 3 is not moved. Provide a formal proof that H is a subgroup of S.

7. Use the provided Cayley graph of a group G to answer the following:



(a) [1] How many generators are used in the Cayley graph?

(b) [2] Simplify $ts^2tst^2st^3sts^2$ to a form on the Cayley graph.

(c) [2] Find the inverse of $ts^2tst^2st^3sts^2$

(d) [2] Find the left cosets of $\langle t \rangle$

(e) [2] Is there a subgroup H in G that is normal in G? Provide some justification for your answer.

8. [8] Recall the center of G is a normal subgroup defined as $Z(G) = \{z \in G | zg = gz \, \forall g \in G\}$. Prove formally that if $G/Z(G)$ is cyclic, then G is abelian.