Take home Key

1) Consider $G = \mathbb{Z}_6(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}_6\}$

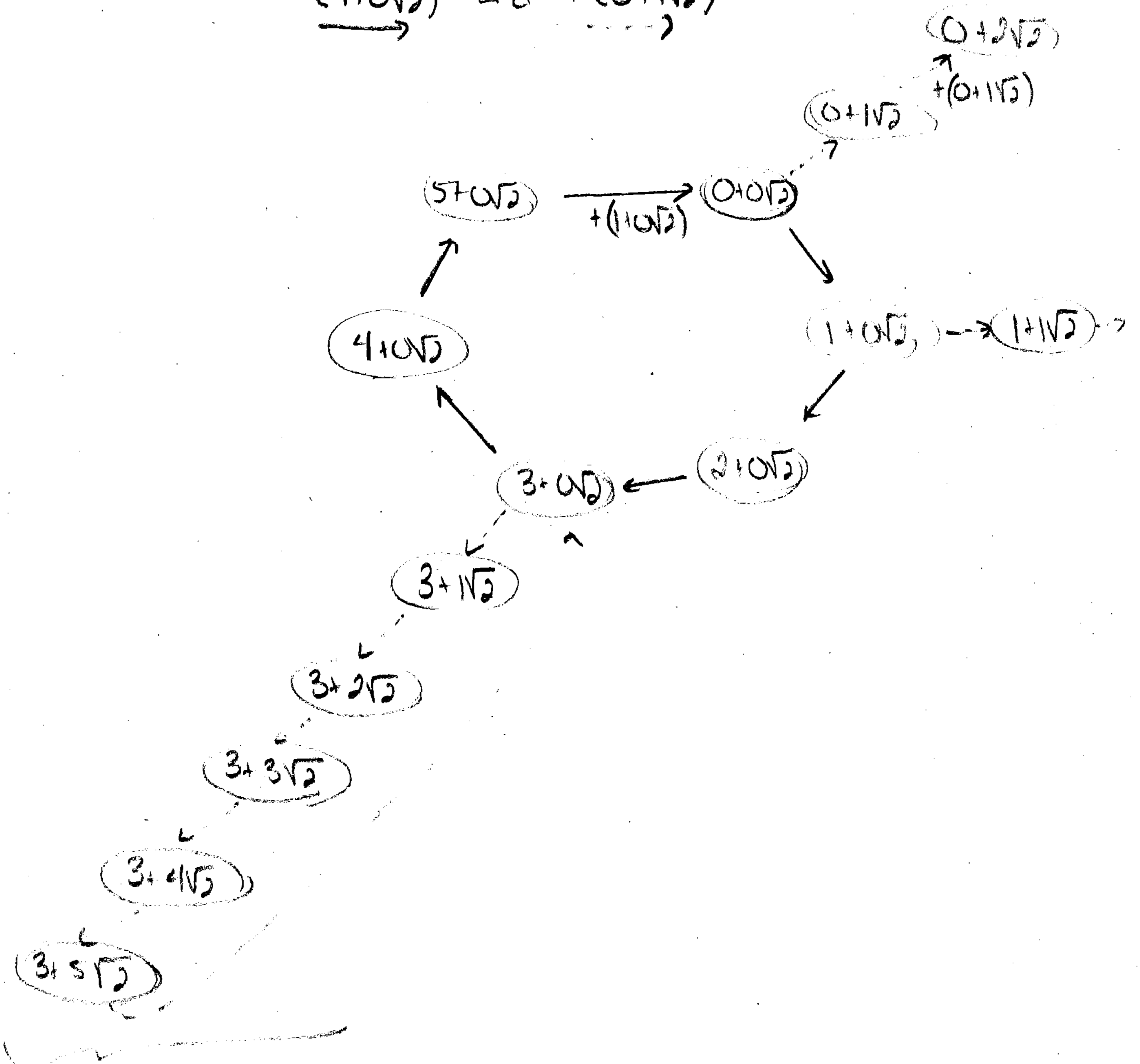
Addition: $(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$ addition mod 6

a) [2] Create a Cayley graph or Cayley Table.

Note there are 36 elements in this group so the Cayley Table is BIG!

Instead: use the generators

$+(1+0\sqrt{2})$ and $+(0+1\sqrt{2})$



this cyclical structure on each corner of hexagon

b) [3] Create a subgroup lattice:

(+1) subgroups arranged by size
(+1) contain most edges
(+.5) found subgroups
(+.5) one diagonal subgroup.

c) [2] If we define $(a_1+b_1\sqrt{2})(a_2+b_2\sqrt{2}) = (a_1a_2+2b_1b_2) + (a_2b_1+a_1b_2)\sqrt{2}$ with multiplication mod 6, what kind of algebraic structure does $\mathbb{Z}_6(\sqrt{2})$ have?

~~cyclic~~ abelian group? yes to abelian group

Commutative ring with unity? yes mult. & add work well together
$1+0\sqrt{2}$ is the mult. identity

field? No, notice $2+0\sqrt{2}$ has no multiplicative inverse

$$(2+0\sqrt{2})(a+b\sqrt{2}) = 2a + 2b\sqrt{2}$$

if $2a+2b\sqrt{2} = 1+0\sqrt{2} \Rightarrow 2\cdot a = 1$ but no such element exists in $\mathbb{Z}_6$

2. [3] The real numbers $\mathbb{R}$ form an additive group and the non-zero real numbers $(\mathbb{R}-\{0\})$ form a multiplicative group.

Consider $\phi: \mathbb{R} \to \mathbb{R}-\{0\}$

$$x \longmapsto e^x$$

(1) Show ϕ is a homomorphism,

Let $x, y \in \mathbb{R}$ then $\phi(x+y) = e^{x+y} = e^x e^y = \phi(x)\phi(y)$ ✓

(1) $\operatorname{Ker}\phi: \{x \in \mathbb{R} \mid \phi(x) = 1\}$

$= \{x \in \mathbb{R} \mid e^x = 1\} = \{0\}$

(1) $\operatorname{im}\phi: \{\alpha \in \mathbb{R}-\{0\} \mid \exists\, x \in \mathbb{R} \ni \phi(x) = \alpha\}$

Note if $\alpha \in \mathbb{R}-\{0\}$ and $\alpha < 0$, $\nexists\, x \in \mathbb{R} \ni \phi(x) = \alpha$

If $\beta \in \mathbb{R}-\{0\}$ and $\beta > 0$, consider $\ln(\beta)$.

Note $\ln(\beta)$ is defined in $\mathbb{R}$,

Examine $\phi(\ln(\beta)) = e^{\ln(\beta)} = \beta \therefore \beta \in \operatorname{im}\phi$.