Requested Proof

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Theorem . Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_m \in S_n$ be the produce of disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of the cycles σ_1, σ_2, ... and σ_m.

Proof. Let l_i be the length of the cycle σ_i which was defined in the construction of σ above. We will show the order of σ is a multiple of each of the l_i's. Since the least common multiple of the l_i's is the smallest multiplier, it must equal the order of σ. Let β denote the order of σ.

Since β is the order of σ, we know $\sigma^\beta = ()$. We will expand σ^β but first note that σ_i and σ_j are all disjoint from each other so we know the cycles can commute with each other, or more symbolically, that $\sigma_i\sigma_j = \sigma_j\sigma_i$. Then,

$$
\begin{aligned}
() &= \sigma^\beta \\
&= (\sigma_1\sigma_2 \cdots \sigma_m)^\beta \\
&= (\sigma_1\sigma_2 \cdots \sigma_m) \cdots (\sigma_1\sigma_2 \cdots \sigma_m) \\
&= \sigma_1^\beta\sigma_2^\beta \cdots \sigma_m^\beta.
\end{aligned}
$$

This implies that $\sigma_i^\beta = ()$ so β must be a multiple of l_i for all $i \in \{1, 2, 3...n\}$. Thus β must be a multiple of each of the l_i's. Recall the order of σ is the smallest integer such that $\sigma^\beta = ()$, thus β is the smallest multiple of all the l_i's. Thus β is the least common multiple of the l_i's which is what we wanted to show.

□