Median 80%
Mean 73%

Key

Vanderpool

TMath 402

Winter 2024

True/False: If the statement is false, give a counterexample or a brief explanation.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] Let G be a group. If G is abelian, then $g = g^{-1}$ for all $g \in G$.

(+.5) False

Consider $C_4 = \langle r \rangle$
Notice $r(r^3) = r^4 = e$
so $r^3 = r^{-1}$ but $r \neq r^3$.

recall G is abelian if $ab = ba$ $\forall a, b \in G$

(+.5) start
(+.5) def of abelian
(+1.5) logic
eg +.5 direction

2. [3] Let G be a group. If all $g \in G$, $g = g^{-1}$ then G is abelian.

(+.5) True

Let $g, h \in G$. We want to show $gh = hg$.
Consider gh. Because $gh \in G$ we know $gh = (gh)^{-1}$.
Since $(gh)^{-1} = h^{-1}g^{-1}$ (socks & shoes theorem)
We have $gh = h^{-1}g^{-1}$.
Since $h = h^{-1}$ and $g = g^{-1}$ we know $gh = hg$ ✓

(+.5) start
(+.5) def of abelian
(+1.5) logic
eg +.5 direction

3. [3] The group $\mathbb{Z}_9 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_6$

(+.5) False Notice $\mathbb{Z}_9 \times \mathbb{Z}_2$ is cyclic (generated by $(1,1)$).

However $\mathbb{Z}_3 \times \mathbb{Z}_6$ is not! The elements are checked:

(.5) start
(+.5) def/idea of isomorphic
(+.5) use an invariant or thm
(+1) say true things/on topic

order of $(x, 0) \leq 3$
order of $(0, y) \leq 6$
we have diagonals left to check

element	order
$(1,1)$	6
$(1,2)$	3
$(1,3)$	6
$(1,4)$	3
$(1,5)$	6

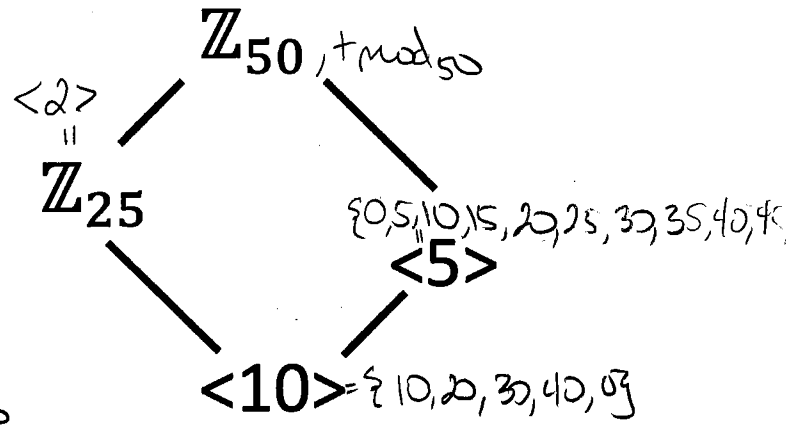
element	order
$(2,1)$	6
$(2,2)$	3
$(2,3)$	6
$(2,4)$	3
$(2,5)$	6

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

OR $\mathbb{Z}_9 \times \mathbb{Z}_2 \cong \mathbb{Z}_{18}$ b/c $\gcd(9,2) = 1$
and $\mathbb{Z}_3 \times \mathbb{Z}_6 \not\cong \mathbb{Z}_{18}$ b/c $\gcd(6,3) \neq 1$ by Fern's Thm
So $\mathbb{Z}_9 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_3 \times \mathbb{Z}_6$

4. [3] The subgroup lattice shown on the right is incorrect. Identify at least two errors.

Stat (1.5)
sense/notation (1.5)



(1.5) • missing the trivial subgroup $\{0\} = \langle 0 \rangle$

(1.5) • missing $\langle 25 \rangle = \{0, 25\}$

Correct (1.5)

• note if you list $\mathbb{Z}_{25}$ & $\mathbb{Z}_{50}$ thats ok too as it should be $\mathbb{Z}_{25} \cong \langle 2 \rangle \leq \mathbb{Z}_{50}$

5. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

using same notation as written HW (4)
stat (1.5)
definitions (1)
notation (1)
approach (1.5)
logic (1)

Theorem 1. *Let G be a group. Prove that an element can only appear once in each column of a Cayley table of G.*

Theorem 2. *Let G be the group generated by the primitive second root of unity, ζ_2, and the primitive third root of unity ζ_3. (In the complex plane $\zeta_2 = e^{\pi i}$ and $\zeta_3 = e^{\frac{2\pi i}{3}}$.) Prove that G is cyclic.*

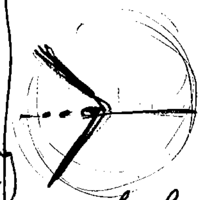
Proof of Thm 1:

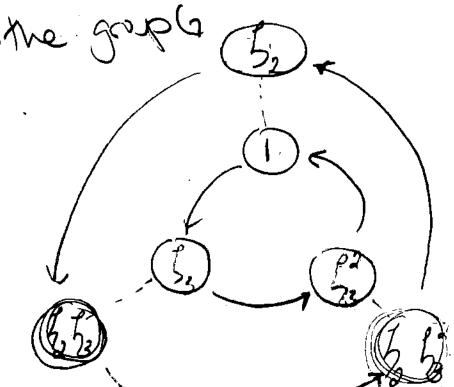
Assume towards a contradiction that an element g appears twice in a column of a Cayley table of a group G.
Let h be the element in G corresponding to this column in the Cayley table.

Since g appears twice, there is a x_1 and x_2 in G corresponding to the 2 rows with g in the same col.

By construction of the Cayley table, this means $x_1 h = g$ and $x_2 h = g$.
So (by transitivity) $x_1 h = x_2 h$.
Since G is a group, $h^{-1} \in G$ so $x_1 h h^{-1} = x_2 h h^{-1} \Rightarrow x_1 e = x_2 e$ or $x_1 = x_2$.
Thus g must have appeared in the same row after all !

Proof of Thm 2
We first generate the group G



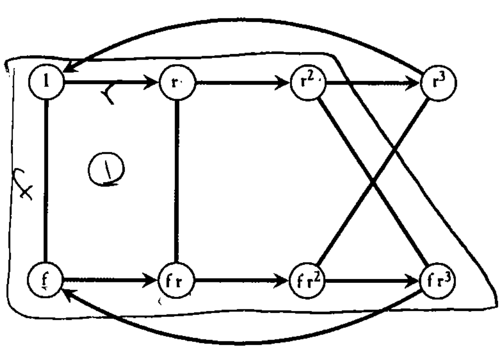


note $\zeta_2 \cdot \zeta_3 = e^{\pi i} e^{\frac{2\pi i}{3}}$
$= e^{\pi i + \frac{2\pi i}{3}}$
$= e^{\frac{2\pi i}{3} + \pi i} = e^{\frac{2\pi i}{3}} e^{\pi i} = \zeta_3 \zeta_2$ commutes!

We are now looking at a problem like #7e (which can be done a few dif. ways)
This group looks like $\mathbb{Z}_3 \times \mathbb{Z}_2$
which is $\cong$ to $\mathbb{Z}_6$ by Fern's Thm.
$\Rightarrow$ we should be able to find a generator.

Try $\langle \zeta_2 \zeta_3 \rangle = \{\zeta_2\zeta_3, \zeta_3^2, \zeta_2, \zeta_3, \zeta_2\zeta_3^2, 1\}$
$= \langle \zeta_2, \zeta_3 \rangle$ //

6. [9] For each of the sets and operations described below, determine if they define a group. If no, briefly explain why. If yes, briefly describe the process you used to reach that answer.

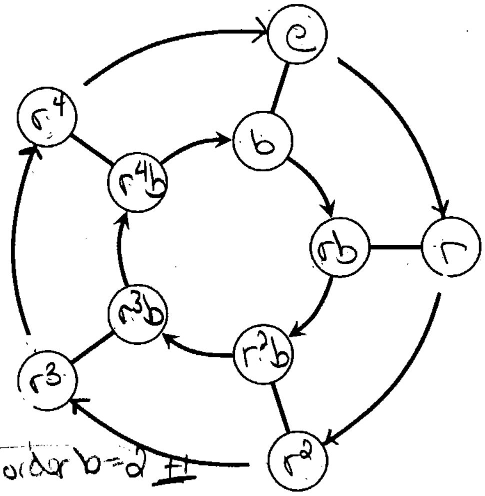
Sets S & Operator $\star$	Is a Group?
a) $\mathbb{Z}$ $\star$ is multiplication .5 +1.5 group prop +1 got it	No We lack inverses for some elements. +.5 Note 1 is the identity. +.5 $\nexists x \in \mathbb{Z} \ni 2 \cdot x = 1.$
(P_h, n_h) (P_h, n_T) $(n_h\ P_h)$ (n_h, P_T) (P_T, n_h) (P_T, n_T) $(n_T\ P_h)$ (n_T, P_T) b) A penny and a nickel are next to each other. We track whether each coin is heads or tails up. The set, is the possible configurations of the penny and nickel. The operations are all that can be done by: *interchange the locations of the two coins or *flip the coin on the right. +1.5 group prop +1 got it P_h penny with heads up n_T nickel with tails up	.5 Yes The identity is "do nothing" Closure: we are given a generating set so we'll build in closure to our set ✓ Inverses: The generators can be undone & any combo can be undone w/ Socks & Shoes Thm.
c) G  +1 analysis +1.5 check reasonable properties	looks a lot like D_n Notice square 1 $\Rightarrow fr = rf$ $(*)$ $\Rightarrow G$ is abelian Highlighted shape $\Rightarrow r^2 f = f r^3$ or $\underline{rf} = f r r r$ using $*$ on underlined sections $\underline{rf}r = f r r r$ $\Rightarrow f r r = f r r r$ applying r^{-1} on the right, 2x $f r r r^{-1} r^{-1} = f r r r r^{-1} r^{-1}$ $\Rightarrow f = fr$ applying f^{-1} on the left $\Rightarrow 1 = r$ so 1 is identity? .5 Not a group

7. A group G has the Cayley graph shown on the right. Let r depict the red action and b depict the blue. Answer the following questions about G:

(a) [1] What is the order of G?

(+.5) def

10   got it (+.5)



(b) [2] Rewrite $r^3br^3b^2r^3r^2b$ into the form r^ib^j where $i \in \{0, 1, 2, 3, 4\}$ and $j \in \{0, 1\}$.

(+.5) got it

+1 following the paths left to right: r

OR

notice $br = rb$
we can switch order?
$r^3r^3r^3r^2bb^2b = r$
order of $r = 5$ and order $b = 2$ +1

(c) [2] Find the orbit of r.

$\{r^i \mid i \in \mathbb{Z}\} = \{r, r^2, r^3, r^4, e\}$

(+1) def of orbit
(+1) got it

(d) [2] Find a subgroup, H, of G such that the order of H is 2.

(+1) def of subgroup
(+.5) order 2
− (+.5) got one

try $\langle b \rangle = \{b, e\}$
Note $e \in H$ so has identity
$b = b^{-1}$ so inverses are in H
$be = b$ so H is closed

(e) [3] Is G cyclic? Justify your answer.

(+1) def of cyclic
(+.5) def of generators
(+1) logic
(+.5) sense/notation

This looks like $\mathbb{Z}_5 \times \mathbb{Z}_2$
which is isomorphic to $\mathbb{Z}_{10}$
b/c $\gcd(5,2) = 1$ (by Fermi's Thm).
So this group is likely cyclic?.

Try $\langle br \rangle$

$\langle br \rangle = \{br, r^2, r^3b, r^4, b, r, r^2b, r^3, r^4b, e\} = G$
So yes?.

OR

we can try each element in G to see if it generates G.
$|\langle r \rangle| = 5$ ✗
$|\langle b \rangle| = 2$ ✗
$\langle br \rangle = 10$:) Yes

8. [1] What topic did you study for and not see on this exam?

generators vs elements

eg generators commuting
generators being their own inverse
generators having an inverse