Vanderpool

TMath 402

Winter 2024

True/False: If the statement is false, give a counterexample or a brief explanation.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

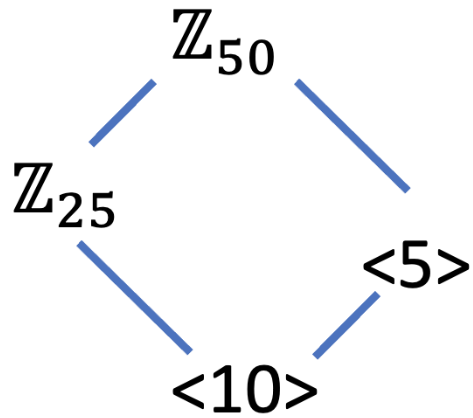
1. [3] Let G be a group. If G is abelian, then $g = g^{-1}$ for all $g \in G$.

2. [3] Let G be a group. If all $g \in G$, $g = g^{-1}$ then G is abelian.

3. [3] The group $\mathbb{Z}_9 \times \mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_6$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. [3] The subgroup lattice shown on the right is incorrect. Identify at least two errors.

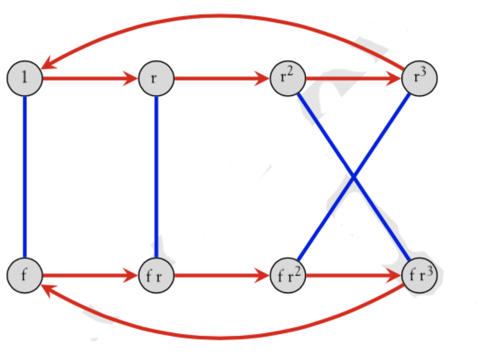


5. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
   No, doing both questions will not earn you extra credit.

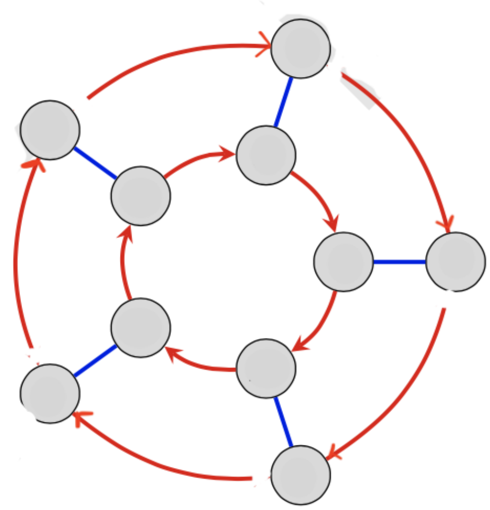
   Theorem 1. *Let G be a group. Prove that an element can only appear once in each column of a Cayley table of G.*

   Theorem 2. *Let G be the group generated by the primitive second root of unity, ζ_2, and the primitive third root of unityζ_3. (In the complex plane $\zeta_2 = e^{\pi i}$ and $\zeta_3 = e^{\frac{2\pi i}{3}}$.) Prove that G is cyclic.*

6. [9] For each of the sets and operations described below, determine if they define a group. If no, briefly explain why. If yes, briefly describe the process you used to reach that answer.

Sets S & Operator $\star$	Is a Group?
a) $\mathbb{Z}$ $\star$ is multiplication	
b) A penny and a nickel are next to each other. We track whether each coin is heads or tails up. The set, is the possible configurations of the penny and nickel. The operations are all that can be done by: *interchange the locations of the two coins or *flip the coin on the right.	
c) 	

7. A group G has the Cayley graph shown on the right. Let r depict the red action and b depict the blue. Answer the following questions about G:



(a) [1] What is the order of G?

(b) [2] Rewrite $r^3br^3b^2r^3r^2b$ into the form r^ib^j where $i \in \{0, 1, 2, 3, 4\}$ and $j \in \{0, 1\}$.

(c) [2] Find the orbit of r.

(d) [2] Find a subgroup, H, of G such that the order of H is 2.

(e) [3] Is G cyclic? Justify your answer.

8. [1] What topic did you study for and not see on this exam?