Take Home Final

TMath 402

Winter 2024

This section is to be taken home, completed, and turned in through Canvas by 8:00pm Tuesday March 12th. There is no time limit and you do not need to type up your solutions to get full marks although the answers should be well edited and readable.

You may discuss this problem with anyone else from the class and use the class resources posted on Canvas. You may not consult anyone or any resource that is not affiliated with the class such as tutors, websites, or other textbooks.

1. Consider $G = \mathbb{Z}_6(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Z}_6\}$. Addition of two elements $a_1 + b_1\sqrt{2}$, $a_2 + b_2\sqrt{2} \in \mathbb{Z}_6(\sqrt{2})$ is defined by: $(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$ where $+$ is addition modulo 6.

   (a) [2] Create a Cayley Graph or Cayley Table for G.

   (b) [3] Create a subgroup lattice for G.

   (c) [2] If we define $(a_1 + b_1\sqrt{2}) \times (a_2 + b_2\sqrt{2}) = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2)\sqrt{2}$ where xy is multiplication modulo 6, what kind of algebraic structure does $\mathbb{Z}_6(\sqrt{2})$ have (cyclic abelian group? commutative ring with unity? field?). Justify your answer.

2. [3] The real numbers, $\mathbb{R}$ form an additive group and the non-zero real numbers, $\mathbb{R} \setminus \{0\}$ form a multiplicative group. Consider the map $\phi : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ given by $\phi(x) = e^x$. Show ϕ is a homomorphism and find its kernel and image.