Format: 60 min individual exam + 2-sided sheet of notes
15 min discussion as group without exams
40 min individual exam - write in a different color

Thur

Final Take 2

TMath 402

Winter 2024

Assessment Look@#7 Bg 4

True/False: If the statement is false, give a counterexample or a brief explanation.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] If the generators of a group G are their own inverses, then $x = x^{-1}$ for all $x \in G$

False (+.5)

start (+.5)

definitions (+.5)

logic (+1)

sense/notation (+.5)

Consider $D_3 = \{e, r, r^2, f, rf, r^2f\}$

$= \langle rf, f \rangle = \{rf, r, r^2, f, r^2f, e\}$

note $(rf)^2 = rfrf = e \Rightarrow (rf)^{-1} = rf$

$f^2 = ff = e \Rightarrow f^{-1} = f$

2. [3] The subgroup $\langle 3, 5\rangle$ in $\mathbb{Z}_{42}$ is cyclic.

True (+.5)

start (+.5)

cyclic/generators def (+.5)

logic (+1)

sense/notation (+.5)

Note $3 + 3 + (-5) = 1 \in \langle 3, 5 \rangle$

Thus $\langle 1 \rangle \subset \langle 3, 5 \rangle$

Since $\langle 1 \rangle = \mathbb{Z}_{42}$

The subgroup $\langle 3, 5 \rangle$ is in fact all of $\mathbb{Z}_{42}$ which is cyclic.

3. [3] All subrings in a ring R are also ideals.

False (+.5)

start (+.5)

def subring/ideals (+.5)

logic (+1)

sense/notation (+.5)

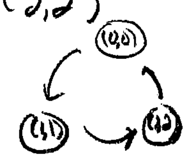
Consider $\mathbb{Z}_3 \times \mathbb{Z}_3$ with coordinant wise addition and mult. mod 3

$\langle (1,1) \rangle = \{(1,1), (2,2), (0,0)\} = S$

is a subgroup: identity $= (0,0)$

inverses $-(1,1) = (2,2)$

closed ✓



is a subring: distributes over addition nicely

closed $(2,2) \cdot (2,2) = (1,1)$

Not an ideal, Consider $(1,0) \cdot (1,1) = (1,0) \notin S$

Let $\mathbb{F}$ be a field with 4 elements

Note $0 \in \mathbb{F}$ b/c $\mathbb{F}$ has an abelian group structure

Note $1 \in \mathbb{F}$ b/c $\mathbb{F}$ has unity.

By def $0 \neq 1$. $\Rightarrow$ We have accounted for 2 elements.

~~$1+1$; if $1+1=0 \Rightarrow \mathbb{F} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as abelian groups~~

Let $x \in \mathbb{F} \ni x \neq 0$ and $x \neq 1$.

Consider $\langle x \rangle$. $o(x) \mid 4$ If $o(x) = 0$ then $x = 0$ $\forall$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. [6] For each below, find a set and binary operator(s) that satisfies the criteria given. If no set exists, explain why. sense/notation (+1.5)

start (+.5)

(a) A group G of order bigger than 2 and where $x = x^{-1}$ for all $x \in G$.

(+.5) (+.5) (+.5)

V_4 works $\quad V_4 = \{v, h \mid v^2 = h^2 = (vh)^2 = e\}$

note $x^2 = e \Rightarrow x = x^{-1} \quad \forall x \in V_4$

(b) A ring without unity.

(+.5) (+.5)

$2\mathbb{Z} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$
with integer addition and multiplication.
Is a subring of $\mathbb{Z}$ but $1 \notin 2\mathbb{Z}$

(+1) (+.5)

(c) A field with 4 elements.

We know all groups with order 4: $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$
Both are abelian so the question is if we can set up a ring structure, where each element is a unit.
Note mult. mod 4 has no mult. inverse for 2
Notice mult, mult 2 has no mult. inverse for (0,1)

5. Consider the map $\phi : D_4 \to \mathbb{Z}_8$ defined by $\phi(rf) = 4$ and $\phi(r) = 4$.

(a) [2] Does ϕ define a homomorphism? Justify your answer.

$D_4 \longrightarrow \mathbb{Z}_8$
$rf \longmapsto 4$
$r \longmapsto 4$
$f \longmapsto 0$
$r^2 \longmapsto 0$
$r^3 \longmapsto 4$
$r^2 f \longmapsto 0$
$e \longmapsto 0$

(+1) Note that $\langle rf, f \rangle = D_4$ (since $rf \cdot r = r r^3 f = f$ & $\langle r, f \rangle = D_4$)
Defining ϕ on generators of D_4 is enough to define a map.

(+1) Homomorphism? By observation of elements on left $\phi(r^i) = i(\phi(r))$
Checking $\phi(r^i f) = \phi(f r^{4-i})$ condition

$4i = i\phi(r) + \phi(f) \qquad \phi(f) + (4-i)\phi(r) = 0 + (4-i)\cdot 4 \mod 8$

$i = 0$ ✓ 1 ✓ 2 ✓ 3 ✓

(b) [3] Find the kernel ($\ker(\phi)$) and image ($\text{im}(\phi)$) of ϕ

(+1) $\ker\phi = \{x \in D_4 \mid \phi(x) = 0 \in \mathbb{Z}_8\} = \{f, r^2, r^2 f, e\} \trianglelefteq D_4$

(+1) $\text{im}\,\phi = \{z \in \mathbb{Z}_8 \mid \exists y \in D_4 \ni \phi(y) = z\} = \{0, 4\} = 4\mathbb{Z}_8 \triangleleft \mathbb{Z}_8$

(c) [2] Identify the elements of $\mathbb{Z}_8/\text{im}(\phi)$.

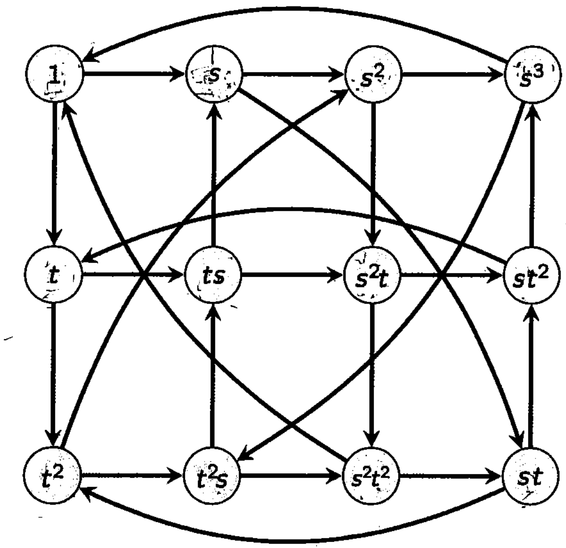
Note $\mathbb{Z}_8$ is abelian $\Rightarrow \text{im}(\phi) \triangleleft \mathbb{Z}_4$ so reasonable question

(+1) $\text{im}\,\phi = \{0, 4\} \qquad 2 +_2 \text{im}\,\phi = \{2, 6\}$
$1 + \text{im}\,\phi = \{1, 5\} \qquad 3 + \text{im}\,\phi = \{3, 7\}$

notation (+.5)

2 element cosets (+.5)

6. A group G has the Cayley graph shown on the right. Answer the following questions about G:



(a) [1] What action does the red arrow represent?

action by s/mult by s on right.

(b) [1] Find $(s^2)^{-1}$.

s^2 b/c $s^2 s^2 = 1$

def of order (+.5)
follow Cayley (+1)

(c) [2] Find the order of ts

$(ts)(ts) = s^2$

$(ts)(ts)(ts) = st^2$

$(ts)(ts)(ts)(ts) = 1$

$\Rightarrow$ 4 (+.5)

(d) [1] Find the orbit of s^3.

$\underbrace{\{s^3, s^2, s, 1\}}_{(+.5)}$

def of orbit (+.5)

(e) [3] Find the right cosets of $H = \{ts, s^2, st^2, e\}$.

coset idea (+.5)
right cosets (+.5)
written as reported (+1) on Cayley graph

$Hs = \{tss, s^2s, st^2s, es\} = \{s^2t, s^3, t, s\}$

$Ht^2 = \{tst^2, s^2t^2, st^2t^2, et^2\} = \{st, s^2t^2, t^2s, t^2\}$

got them (+1) / no ordering

(f) [2] Describe the elements of G/H, if possible. If not, explain why.

(+1) Consider $sH = \{sts, ss^2, sst^2, se\} = \{t^2, s^3, s^2t^2, s\}$
$\neq Hs$

(+.5) The subgroup H is not normal in G
$\Rightarrow$ we cannot make a well defined binary operation on G/H

sense (+.5)

partial 1.5 using coset language

16
4 4
2 2 2 2

Writing Rubric (+4)

44
4 11
2 2

7. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.
The same criteria used for the WrittenHW will be used here, so be sure to clearly indicate your reasoning and do not just perform computations/algebraic manipulations.

notation (+1)

Theorem 1. *Let $\phi : G \to H$ be a group homomorphism, then* $\ker(\phi)$ *is a subgroup of* G.

Theorem 2. *There are only four possible group homomorphisms from* $\mathbb{Z}_{44}$ *to* $\mathbb{Z}_{16}$.

Thm 1 Proof

We will verify the properties of a subgroup:

(+1) $\boxed{\text{Closure}}$: Let $a, b \in Ker(\phi)$. We want to show $ab \in Ker(\phi)$.

Since $a, b \in Ker\,\phi$ we have $\phi(a) = e = \phi(b)$

Consider $\phi(ab) = \phi(a)\phi(b)$ b/c ϕ is a homomorphism

$= e \cdot e$ b/c $a, b \in Ker\,\phi$

$= e$ by def of identity

$\therefore ab \in Ker\,\phi$ & we have closure.

(+1) $\boxed{\text{Identity}}$: Let e_G be the identity in G. Let $g \in G$. Consider

$\phi(g) = \phi(g e_G) = \phi(g)\phi(e_G)$ b/c ϕ is a homomorphism.

Since H is a group $[\phi(g)]^{-1} \in H$

$\Rightarrow [\phi(g)]^{-1}\phi(g) = [\phi(g)]^{-1}\phi(g)\phi(e_G)$

$\Rightarrow e_H = e_H\,\phi(e_G)$

$\Rightarrow \phi(e_G) = e_H \Rightarrow e_G \in Ker\,\phi$.

(+1) $\boxed{\text{Inverses}}$ Let $a \in Ker\,\phi$, we want to show $a^{-1} \in Ker\,\phi$. Consider

$\phi(e_G) = \phi(aa^{-1}) = \phi(a)\,\phi(a^{-1})$ b/c ϕ is homom

$= e_H\,\phi(a^{-1})$ thus

$e_H = e_H\,\phi(a^{-1}) \Rightarrow \phi(a^{-1}) = e_H$

$\Rightarrow a^{-1} \in Ker\,\phi$ So we have inverses.

Thus $Ker\,\phi$ is a subgroup. //

Thm 2 Proof: Let $\phi : \mathbb{Z}_{44} \to \mathbb{Z}_{16}$

Since $Ker\,\phi \leq \mathbb{Z}_{44} \Rightarrow |Ker\,\phi| \mid 44$

$\Rightarrow |Ker\,\phi| = 1, 2, 4, 11, 22, 44$ by LaGrange's Thm

Similarly since $im\,\phi \leq \mathbb{Z}_{16}$, $|im\,\phi| \mid 16$

$\Rightarrow |im\,\phi| = 1, 2, 4, 8, 16$ (+1)

By the fund. isomorphism thm

$\mathbb{Z}_{44}/Ker\,\phi \cong im\,\phi$

$\Rightarrow |\mathbb{Z}_{44}| = 44 = |im\,\phi| \cdot |Ker\,\phi|$ (+1)

So only possibilities are $|im\,\phi| = 1, 2, 4$. We look at each:

$\boxed{\text{Size of image is 1}}$ so $\{0\}$

Define $h : \mathbb{Z}_{44} \to \mathbb{Z}_{16}$, $1 \mapsto 0$ so $h(x) = 0\ \forall x$ "trivial homom"

$\boxed{\text{Size of image is 2}}$ so $\{0, 8\}$

Define $f : \mathbb{Z}_{44} \to \mathbb{Z}_{16}$, $1 \mapsto 8$

Note we need $f(1) \neq 0$ to be distinct from h

$\boxed{\text{Size of image is 4}}$ so $\{0, 4, 8, 12\}$

(+1) Define $g : \mathbb{Z}_{44} \to \mathbb{Z}_{16}$, $1 \mapsto 4$ and $k : \mathbb{Z}_{44} \to \mathbb{Z}_{16}$, $1 \mapsto 12$

Note we need $g(1) \neq 0$ or 8 to be distinct

(+1) We are heavily relying on the fact that defining the image of a generator defines the entire map. Also b/c $im\,\phi \leq \mathbb{Z}_{16}$ we are seriously limited in image choices