Assessment Key

Mon

Final

TMath 402

Look@ #7 B 4

Winter 2024

True/False: If the statement is false, give a counterexample or a brief explanation.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] The group G is cyclic with generator $\langle a \rangle$ and order 20. Then $\langle a^3 \rangle$ also generates G.

True (+.5)

Note $\gcd(3,20)=1 \Rightarrow \exists\, \alpha, \beta \in \mathbb{Z} \ni 3\alpha + 20\beta = 1$

Consider $(a^3)^\alpha = a^{3\alpha} \equiv a^1 \bmod 20$

Thus $\langle a^3 \rangle \subseteq \langle a^1 \rangle = G$

$\therefore \langle a^3 \rangle = G$. So a^3 generates G.

start (+.5)
generator def (+.5)
sense/notation (+.5)
logic (+1)

2. [3] If H is a normal subgroup in G, then $ghg^{-1} = h$ for all $h \in H$ and for all $g \in G$.

False (+.5)

Consider D_5. Note $\langle r \rangle = \{r, r^2, r^3, r^4, e\} \leq D_5$

and b/c $[D_5 : \langle r \rangle] = 2 \Rightarrow \langle r \rangle \trianglelefteq D_5$

However $\underset{g}{f}\,\underset{h}{r}\,\underset{g}{f^{-1}} = frf = r^4 ff = r^4 \neq \underset{h}{r}$

start (+.5)
verify normal (+.5)
example (+1)
sense/notation (+.5)

3. [3] All subgroups in a ring R are also subrings.

False (+.5)

Consider $\mathbb{Z}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$

The set $H = \{0 + b\sqrt{2} \mid b \in \mathbb{Z}\}$ is a subgroup

closure $b\sqrt{2} + \beta\sqrt{2} = (b+\beta)\sqrt{2}$ ✓

inverses $-b\sqrt{2}$ b/c $b\sqrt{2} + (-b\sqrt{2}) = (b-b)\sqrt{2} = 0\sqrt{2}$

identity $0 + 0\sqrt{2} \in H$

Note H is not closed under multiplication

b/c $3\sqrt{2} \cdot \sqrt{2} = 3 \cdot 2 = 6 \notin H$

start (+.5)
def of subring (+.5)
def of subgroup (+.5)
good ex (+1)

9

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. [6] For each below, find a set and binary operator(s) that satisfies the criteria given. If no set exists, explain why.

sense/notation (+.5)
steps (+.5)

note: there are lots of answers

(a) An abelian group of order 8 that is not cyclic.
(+.5) (+.5) (+.5)

$\mathbb{Z}_4 \times \mathbb{Z}_2$ could work. Note $\nexists\, x \in \mathbb{Z}_4 \times \mathbb{Z}_2 \ni |x| = 8$
(addition mod 4, addition mod 2) $\Rightarrow \mathbb{Z}_4 \times \mathbb{Z}_2$ is not cyclic

(b) A ring that is non-commutative.
(+.5) (+.5)

$\mathbb{H} = \{ a + bi + cj + dk \mid a,b,c,d \in \mathbb{Z},\ i,j,k \in Q_8 \}$
addition is component wise
mult is dist / foil where $ij = k$ $ji = -k$ (mult. determined by Q_8)

(c) A ring with unity that is not a field.
(+.5) (+.5) (+.5)

$\mathbb{Z}$ standard addition & multiplication

note $1 \in \mathbb{Z} \Rightarrow$ has unity
note $2 \in \mathbb{Z}$ but has no multiplicative inverse

5. Consider the map $\phi : D_4 \to \mathbb{Z}_4$ defined by $\phi(rf) = 2$ and $\phi(f) = 0$.

(a) [2] Does ϕ define a homomorphism? Justify your answer.

D_4	$\mathbb{Z}_4$
rf	2
f	0
r	2
r^2	0
r^3	2
r^2f	0
r^3f	2
e	0

(+1) Note that $\langle rf, f \rangle = D_4$ [since $rf \cdot f = r$ and we know $\langle r,f \rangle = D_4$]
Defining ϕ on generators of D_4 is enough to define a map.

(+1 check) (check condition)
Homomorphism? $\phi(rf \cdot f) = \phi(rf) + \phi(f) = 2 + 0 = 2$ ✓
$\phi(r^i f)$ vs $\phi(f r^{4-i})$ note $\phi(r^i)$ vs $(\phi(r))^i$ ✓
$\phi(r^i) + \phi(f)$ $\phi(f) + \phi(r^{4-i})$
$\phi(r^i) + 0$ $\phi(r^{4-i})$ works for all i ✓ So, yes

(b) [3] Find the kernel ($\ker(\phi)$) and image ($\mathrm{im}(\phi)$) of ϕ

(+1) $\ker \phi = \{ x \in D_4 \mid \phi(x) = 0 \in \mathbb{Z}_4 \} = \{ f, r^2, r^2 f, e \} \trianglelefteq D_4$

(+1) $\mathrm{im}\, \phi = \{ z \in \mathbb{Z}_4 \mid \exists\, y \in D_4 \ni \phi(y) = z \} = \{0, 2\} = 2\mathbb{Z}_4 \trianglelefteq \mathbb{Z}_4$

definitions of each (+1)

(c) [2] Identify the elements of $\mathbb{Z}_4/\mathrm{im}(\phi)$.

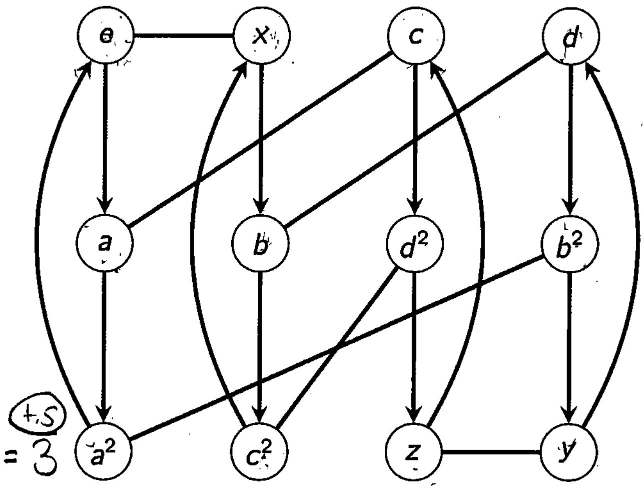
Note $\mathbb{Z}_4$ is abelian $\Rightarrow \mathrm{im}(\phi) \trianglelefteq \mathbb{Z}_4$ so factor groups can work

notation (+.5)

(+1) $\mathrm{im}(\phi) = \{0, 2\}$
$1 + \mathrm{im}(\phi) = \{1, 3\}$
→ 2 elements/cosets (+.5)

13

6. A group G has the Cayley graph shown on the right. Answer the following questions about G:



(a) [1] What action does the red arrow represent?

action by a/mult by a on right

(b) [1] Find x^{-1}. (what sends x back to e)

X

def of order (+.5)
follow Cayley (+1)

(c) [2] Find the order of c = ax

(ax)(ax) = c^2
(ax)(ax)(ax) = e => order of c = 3 (+.5)

(d) [1] Find the orbit of a^2.

$\underbrace{\{a^2, a, e\}}_{(+.5)}$

def of orbit (+.5)

coset idea (+.5)
right coset (+.5)
written as reported (+1) on Cayley graph

(e) [3] Find the right cosets of $H = \{b, b^2, e\}$. b = xa , c = ax

Ha = {ba, b²a, ea} = {xaa, xaxaa, a} = {c², y, a}
Ha² = {ba², b²a², ea²} = {xaaa, xaxaaa, a²} = {x, d, a²}
Hc = {bc, b²c, ec} = {xaax, xaxaax, c} = {d², z, c} = Hz = Hd²
→ got here (+1)

(f) [2] Describe the elements of G/H, if possible. If not, explain why.

(+1) Consider aH = {ab, ab², ae} = {axa, axaxa, a} = {d², x, a} ≠ Ha

(+.5) The subgroup H is not normal in G
=> we can not make a well defined binary operator on G/H

sense (+.5)

10

28
2 14
2 7

60
2 30
2 15
3 5

7. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.
The same criteria used for the WrittenHW will be used here, so be sure to clearly indicate your reasoning and do not just perform computations/algebraic manipulations.

Theorem 1. *Let H be a subgroup of G and $|H| > 1$, then $aH = bH$ for all $b \in aH$.*

Theorem 2. *There are only four possible group homomorphisms from $\mathbb{Z}_{60}$ to $\mathbb{Z}_{28}$.*

Thm 1 Proof:
We will show for $b \in aH$ that $aH \subseteq bH$ AND $bH \subseteq aH$.
Double containment $\Rightarrow aH = bH$.
Let $b \in aH$. Then $\exists\, h \in H$ so that $b = ah$, or equivalently $bh^{-1} = a$.
$aH \subseteq bH$: Let $\alpha \in aH$. Then $\exists\, h_1 \in H \ni \alpha = ah_1$. Since $a = bh^{-1}$
$\Rightarrow \alpha = (bh^{-1})h_1 = b(h^{-1}h_1)$.
Recall H is closed $\therefore h^{-1}h_1 \in H$
So $\alpha \in bH$.
Thus $aH \subseteq bH$.
$bH \subseteq aH$: Let $\beta \in bH$. Then $\exists\, h_2 \in H \ni \beta = bh_2$. Since $b = ah$
$\Rightarrow \beta = (ah)h_2 = a(hh_2)$.
Recall H is closed $\therefore hh_2 \in H$
So $\beta \in aH$
Thus $bH \subseteq aH$.
We showed double containment
Thus $aH = bH$ for $b \in aH$. //

Writing Rubric (+4)
logic/double containment (+1)
Coset def (+1)
notation (+1)
subgroup properties (+1)

Thm 2 Proof: let $\phi: \mathbb{Z}_{60} \to \mathbb{Z}_{28}$.
Since $\operatorname{Ker}\phi \leq \mathbb{Z}_{60} \Rightarrow |\operatorname{Ker}\phi| \mid 60$
$\Rightarrow |\operatorname{Ker}\phi| = 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$
(by LaGrange's Thm) (+1)
Similarly, since $\operatorname{im}\phi \leq \mathbb{Z}_{28}$ $|\operatorname{im}\phi| \mid 28$
$\Rightarrow |\operatorname{im}\phi| = 1, 2, 4, 7, 14, 28$.
By the first isomorphism thm:
$\mathbb{Z}_{60}/\operatorname{Ker}\phi \cong \operatorname{im}\phi$ (+1)
$\Rightarrow |\mathbb{Z}_{60}| = 60 = |\operatorname{im}\phi| \cdot |\operatorname{Ker}\phi|$
So the only possibilities are $|\operatorname{im}\phi| = 1, 2,$ or 4. We look at each:

$\boxed{\text{Size of image is 1}}$ so $\{0\}$
Define $h: \mathbb{Z}_{60} \to \mathbb{Z}_{28}$, $1 \mapsto 0$ so $h(x) = 0\ \forall x$ and is a homom. ✓
$\Rightarrow \operatorname{im}(h) = 1$

$\boxed{\text{Size of image is 2}}$ so $\{0, 14\}$
Define $f: \mathbb{Z}_{60} \to \mathbb{Z}_{28}$, $1 \mapsto 14$
note we need to send 1 to nonzero element to distinguish from h above. ✓

$\boxed{\text{Size of image is 4}}$ so $\{0, 7, 14, 21\}$
Define $g: \mathbb{Z}_{60} \to \mathbb{Z}_{28}$, $1 \mapsto 7$ + $k: \mathbb{Z}_{60} \to \mathbb{Z}_{28}$, $1 \mapsto 21$ (+1)

(+1) Recall mapping the generator is enough to define a map. We need to distinguish from h + f above $\Rightarrow$ only $g(1) = 7$ and $k(1) = 21$ are remaining possibilities?

4

1/8

~~9~~	1
1~~3~~	2~~2~~
10	1~~8~~
8	40