Final

TMath 402

Winter 2024

True/False: If the statement is false, give a counterexample or a brief explanation.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] The group G is cyclic with generator $\langle a \rangle$ and order 20. Then $\langle a^3 \rangle$ also generates G.

2. [3] If H is a normal subgroup in G, then $ghg^{-1} = h$ for all $h \in H$ and for all $g \in G$.

3. [3] All subgroups in a ring R are also subrings.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. [6] For each below, find a set and binary operator(s) that satisfies the criteria given. If no set exists, explain why.

   (a) An abelian group of order 8 that is not cyclic.

   (b) A ring that is non-commutative.

   (c) A ring with unity that is not a field.

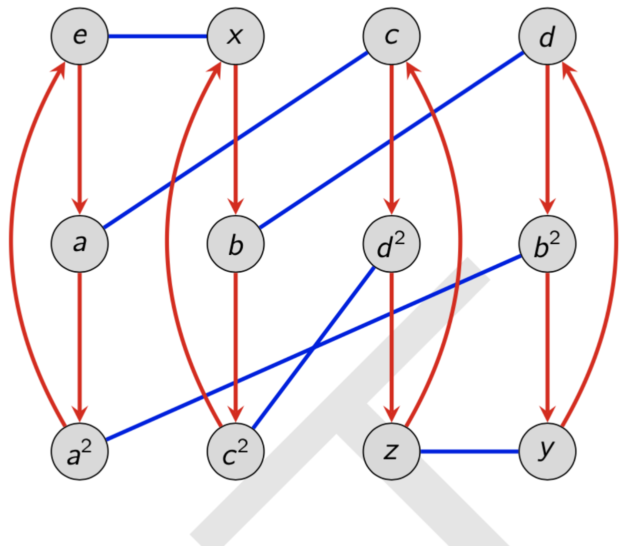
5. Consider the map $\phi : D_4 \to \mathbb{Z}_4$ defined by $\phi(rf) = 2$ and $\phi(f) = 0$.

   (a) [2] Does ϕ define a homomorphism? Justify your answer.

   (b) [3] Find the kernel ($\ker(\phi)$) and image ($\text{im}(\phi)$) of ϕ

   (c) [2] Identify the elements of $\mathbb{Z}_4/\text{im}(\phi)$.

6. A group G has the Cayley graph shown on the right. Answer the following questions about G:



(a) [1] What action does the red arrow represent?

(b) [1] Find x^{-1}.

(c) [2] Find the order of c

(d) [1] Find the orbit of a^2.

(e) [3] Find the right cosets of $H = \{b, b^2, e\}$.

(f) [2] Describe the elements of G/H, if possible. If not, explain why.

7. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
   No, doing both questions will not earn you extra credit.
   The same criteria used for the WrittenHW will be used here, so be sure to clearly indicate your reasoning and do not just perform computations/algebraic manipulations.

   Theorem 1. *Let H be a subgroup of G and $|H| > 1$, then $aH = bH$ for all $b \in aH$.*

   Theorem 2. *There are only four possible group homomorphisms from $\mathbb{Z}_{60}$ to $\mathbb{Z}_{28}$.*