Final

TMath 402

Winter 2016

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] (§1) Let D_4 be the dihedral group of order 8. For all a, b, and c in D_4, if $ab = bc$, then $a = c$.

2. [3] (§4) Let G be a group and $G = \langle x \rangle$ If g, $h \in G$, then there exists $m \in \mathbb{Z}$ such that $x^m g = h$.

3. [3] (§9) If H is a normal subgroup of a group G, then $ghg^{-1} = h$ for all $h \in H$ and $g \in G$.

4. [3] (§13) Let a and b belong to a commutative ring. If ab is a zero divisor, then a or b is a zero divisor.

5. (§2, 12 & 13) For each below, find a set and binary operator(s) that satisfies the criteria given, if one exists. If no set exists, explain why.

   (a) [3] a ring, but not a field

   (b) [3] an integral domain, but not a ring

   (c) [3] a ring, but not a group

   (d) [3] a group but not an abelian group

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.
(If you use a calculator, be sure to tell me.)

6. Consider S_5, the group of permutations on five elements.

   (a) [2] (§5 & 3) Let $H = \{\sigma \in S_5 | \sigma(5) = 1\}$. Determine if $H \leq S_5$ and justify your answer.

   (b) [4] (§5, 7, & 3) Define a subgroup N of S_5 such that $[S_5 : N] = 6$. Justify your choices.

   (c) [3] (§0) Note that S_5 is itself a set (the set of permutations on five elements). Devise an equivalence relation on S_5 and justify your answer.

7. Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_6$ under addition modulo 2 in the first coordinate and addition modulo 3 in the second.

   (a) [1] (§2) Find $|G|$.

   (b) [4] (§3) Write down the subgroup lattice for G

   (c) [2] (§7) Identify the distinct cosets of $\langle(0,2)\rangle$ in G.

   (d) [4] (§6 & 9) What group is $\mathbb{Z}_2 \oplus \mathbb{Z}_6/\langle(0,2)\rangle$ isomorphic to? Prove it.

8. Consider the set of polynomials with real coefficients $\mathcal{P}$ with the binary operator of addition. Define a map ϕ that takes a polynomial to its derivative.

   (a) [3] Determine if ϕ is a homomorphism, an isomorphism, or an automorphism. Prove your conclusion.

   (b) [3] (§10) Identify $\ker \phi$.

9. [5] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
   No, doing both questions will not earn you extra credit.

   (a) (§7) Suppose G is a finite group of order n and m is relatively prime to n. If $g \in G$ and $g^m = e$, prove that $g = e$.

   (b) (§6) Find the order of $\text{Aut}(\mathbb{Z}_n)$ where n is a fixed positive integer. Prove your result.