Fx #2

Key

FINAL **TMath 402** Practice

Note: This is a practice final and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

True/False: If the statement is *always* true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. The relation R on the set $\mathbb{Z}$ defined below is an equivalence relation.

$$R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} | 5 \text{ divides } a-b\}.$$

True

Reflective: notice $5 \cdot 0 = x - x \Rightarrow 5 | (x-x) \Rightarrow (x,x) \in R$

Symmetric: if $(x,y) \in R \Rightarrow 5|(x-y) \Rightarrow 5d = x-y$ for some $d \in \mathbb{Z}$
$\Rightarrow -5d = -(x-y) = -5d = y-x \Rightarrow 5|(y-x) \Rightarrow (y,x) \in R$.

Transitive: if $(x,y),(y,z) \in R \Rightarrow 5d = x-y$ & $5c = y-z$ for $d,c \in \mathbb{Z}$
$\Rightarrow 5d + 5c = x-y+y-z \Rightarrow 5(d+c) = x-z \Rightarrow (x,z) \in R$ //

Notice this is a sentence fragment. I've modified the question so it makes sense.

2. The set S of all pairs (x,y) of real numbers such that $y \neq 0$, under the operation $(x,y) \star (z,w) = (x+z, yw)$, is a binary operator.

True: Closure, if $(x,y),(z,w) \in S$ then $(x,y) * (z,w) = (x+z, yw) \in S$ b/c neither y nor w were equal to zero & in $\mathbb{R}$ we have the zero multiplicative property.

Associative: let $(x,y),(z,w),(a,b) \in S$ then b/c assoc of $\mathbb{R}$
$[(x,y)*(z,w)]*(a,b) = (x+z, yw)*(a,b) = (x+z+a, ywb) = (x,y)*(z+a, wb) = (x,y)*[(z,w)*(a,b)]$

3. If G is a group in which $(ab)^2 = a^2b^2$ for all $a \in G$ and $b \in G$, then G is abelian.

True: Let $a,b \in G$ then $(ab)^2 = a^2b^2 \Rightarrow (ab)(ab) = a^2b^2$
which implies $a^{-1}abab = a^{-1}aabb \Rightarrow bab = abb$
applying b^{-1} on the right $\Rightarrow babb^{-1} = abb^{-1}$
$\Rightarrow ba = ab$. Since a & b were arbitrary G is abelian.

4. If H is a normal subgroup of a group G, then $ghg^{-1} = h$ for all $h \in H$ and $g \in G$.

False: Consider D_4. The set $\{r, r^2, r^3, e\}$ is a normal subgp of G (b/c closed under op, inverses & has index 2) but

$frf^{-1} = frf = ffr^3 = r^3 \neq r$

5. If H and K are subgroups of G, $H \cong K$, then $G/H \cong G/K$.

False Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ with $H = \mathbb{Z}_2 \times \{0\}$ and $K = \{0\} \times \langle 2 \rangle$

Notice $H \cong \mathbb{Z}_2 \cong K$ but

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathbb{Z}_4 \\ G/H \arrow[ur, "\cong"'] & \end{tikzcd} \qquad (a,b) \mapsto b \qquad \left. \right\} \; G/H \cong \mathbb{Z}_4 \text{ by 1st iso thm}$$

and

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathbb{Z}_2 \times \mathbb{Z}_2 \\ G/K \arrow[ur, "\cong"'] & \end{tikzcd} \qquad (a,b) \mapsto (a \bmod 2, b \bmod 2)$$

$G/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

6. The group $\mathbb{Z}_{10} \times \mathbb{Z}_{85}$ is cyclic.

False let $(a,b) \in \mathbb{Z}_{10} \times \mathbb{Z}_{85}$.

Recall $o(a) = \dfrac{10}{\gcd(a,10)}$, $o(b) = \dfrac{85}{\gcd(b,85)}$ and so

10 = 2 · 5   85 = 5 · 17

$o(a,b) = \text{lcm}\left(\dfrac{10}{\gcd(a,10)}, \dfrac{85}{\gcd(b,85)}\right)$, even if $\gcd(a,10) = 1 = \gcd(b,85)$

the lcm would be $2 \cdot 5 \cdot 17 = 170$ (the largest order possible for (a,b)

but $|\mathbb{Z}_{10} \times \mathbb{Z}_{85}| = 2 \cdot 5 \cdot 5 \cdot 17 = 850$

7. All rings have an abelian group structure.

True the abelian group structure is required in the definition of a ring for the first binary operator.

8. The element $(2,3)$ is a zero divisor in the ring $\mathbb{Z}_4 \times \mathbb{Z}_6$.

True $\quad (2,3) \cdot (2,2) = (2 \cdot 2, 3 \cdot 2) \equiv (0,0)$

Typo #9

9. For each of the algebraic structures, list *all* properties that it is guaranteed to satisfy:

Algebraic Structures	Properties
F C A a) vector space	A) the set is closed under addition
D B b) group (assume binary operation is multiplicative)	B) the set is closed under multiplication
D B c) quotient/factor group (assume binary operation is multiplicative)	C) there is a 0 (additive identity)
E D B d) cyclic group (assume binary operation is multiplicative)	D) there is a 1 (multiplicative identity)
D B e) normal subgroup (assume binary operation is multiplicative)	E) multiplication is commutative
F B A f) ring	F) addition is commutative
F D C B A g) division ring	G) cancellation laws hold (for all binary operations)

Free Responce: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.
(If you use a calculator, be sure to tell me.)

10. For a group G and an element of the group g, which is larger: the order of the group or the order of the element? When are they equal?

$o(g) \le |G|$ primarily b/c of multiplicative closure. $g^n \in G$ $\forall n$.

$o(g) = |G|$ when G is a cyclic group
and g is a generator. (there may be more than 1 generator.)

11. Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 8 & 5 & 6 & 4 & 7 & 1 \end{pmatrix}$ from S_8 as:

(a) a product of disjoint cycles;

$(1,2,3,8)(4,5,6)(7)$

(b) a product of transpositions.

$(1,8)(1,3)(1,2)(4,6)(4,5)$

(c) Compute σ^4. $[(1,2,3,8)(4,5,6)(7)]^4$

$= (1,2,3,8)^4(4,5,6)^4(7)$

$= (1)(2)(3)(8)\,(4,5,6)^3(4,5,6)(7)$

$= (4,5,6)^3(4,5,6)$

$= (4,5,6)$

not moving

(d) Find the order of σ.

$o(\sigma) = \text{lcm}(o(1,2,3,8), o(4,5,6))$

$= \text{lcm}(4,3) = 12$

(e) Write down σ^{-1}.

$\sigma^{-1} = (1,8,3,2)(4,6,5)$

I should add another interesting question here involving centralizers, or normalizers --

add op on #12
and make one actually a homomorphism.

12. Classify each of the following maps as either: not a homomorphism, homomorphism, monomorphism, epimorphism, isomorphism, or automorphism. Prove your classification.

$(\mathbb{Z}_6, \oplus)$
$\mathbb{Z}_6 \xrightarrow{\varphi} (\mathbb{Z}_{18}, \oplus)$
$x \mapsto 3x+5$

not homomorphism note:
welldefined: if $a \equiv b \bmod 6$, then $\exists d \in \mathbb{Z}$
$\Rightarrow a = b + 6d$ and $\varphi(b) = 3b + 5$
whereas $\varphi(a) = \varphi(b+6d) = 3(b+6d)+5$
Notice $\varphi(a) = 3b + 18d + 5 \equiv 3a+5 \bmod 18$
So $\varphi(a) = \varphi(b)$.
homomorphism: $\varphi(a)\varphi(b) = (3a+5)(3b+5)$
$= 9ab + 15a + 15b + 25$
$\not\equiv 3ab + 5 \bmod 18$
$= \varphi(ab)$

consider $a = 1$ & $b = 2$
$\varphi(ab) = 14$
$\varphi(a)\varphi(b) = 8 \cdot 11$
$\equiv 16 \bmod 18$

$(\mathbb{Q}, +) \xrightarrow{\chi} (\mathbb{Q}, +)$
$\frac{m}{n} \mapsto \frac{m^2}{n^2+1}$

not a well defined map. Notice
$\frac{1}{2} = \frac{2}{4}$ in $\mathbb{Q}$ but
$\chi(\frac{1}{2}) = \frac{1}{5} \neq \frac{4}{17} = \chi(\frac{2}{4})$

13. Groups D_4 and Q_8 have 8 elements. Show that they are not isomorphic.

We could write down their Cayley tables...
or We could consider the order of elements since if $\varphi: D_4 \to Q_8$ was an isomorphism $o(x) = o(\varphi(x))$.
Notice Q_8 has 3 elements of order 4 whereas D_4 has only 2.
or We could compare the lattice of subgroups since if $\varphi: D_4 \to Q_8$ was an isomorphism there would be a 1-1 & onto correspondence between the lattices.

14. Suppose that H is a subgroup of G. Show that the intersection of all conjugate subgroups of H, $\bigcap_{x \in G} xHx^{-1}$, forms a normal subgroup of G.

Subgp: Notice $e \in H$ and $xex^{-1} = e$ $\forall x \in G \Rightarrow e \in \bigcap_{x \in G} xHx^{-1}$
so this is a nonempty set.
(closure) Let $a, b \in \bigcap_{x \in G} xHx^{-1} \Rightarrow a, b \in xHx^{-1}$ $\forall x \in G$
$\Rightarrow \exists h_x, g_x \in H \ni xh_xx^{-1} = a$ and $xg_xx^{-1} = b$, so
$ab = (xh_xx^{-1})(xg_xx^{-1}) = xh_xg_xx^{-1}$ & b/c $H \leq G$ $h_xg_x \in H$
$\Rightarrow xh_xg_xx^{-1} \in xHx^{-1}$. Since this holds $\forall x \in G \Rightarrow ab \in \bigcap_{x \in G} xHx^{-1}$
(inverses) Let $a \in \bigcap_{x \in G} xHx^{-1} \Rightarrow a \in xHx^{-1}$ $\forall x \in G \Rightarrow \exists h_x \in H \ni$
$a = xh_xx^{-1}$. Since $H \leq G$, $h_x^{-1} \in H$. Consider $xh_x^{-1}x^{-1}$;
$a\,xh_x^{-1}x^{-1} = (xh_xx^{-1})(xh_x^{-1}x^{-1}) = xh_xh_x^{-1}x^{-1} = e$, similarly $xh_x^{-1}x^{-1}a = e$
$\Rightarrow xh_x^{-1}x^{-1} = a^{-1}$. Since x was arbitrary $a^{-1} \in \bigcap_{x \in G} xHx^{-1}$.
Normal: let $a \in \bigcap_{x \in G} xHx^{-1}$, then for each $x_i \in G$ $\exists h_{x_i} \in H$ so that $a = x_ih_{x_i}x_i^{-1}$
$\Rightarrow gag^{-1} = gx_ih_{x_i}x_i^{-1}g^{-1} = (gx_i)h_{x_i}(gx_i)^{-1}$ $\forall x_i \in G$ so $gag^{-1} \in \bigcap_{gx \in G}(gx)H(gx)^{-1} = \bigcap_{x \in G} xHx^{-1}$

Sorry for not having this actually written out...

15. Let $G = S_4$ and H be something non trivial that makes sense. Define $\phi : G \to H$...

(a) Create the subgroup lattice of G.

(b) Find $\ker(\phi)$.

$\ker\phi = \{ g \in S_4 \mid \phi(g) = e_H \}$

(c) The factor group $G/\ker(\phi)$ is isomorphic to a familiar group talked about in this course. Find out what it is and then build an isomorphism.

(d) Draw the subgroup lattice of $G/\ker(\phi)$.