Take Home Final

TMath 402

Autumn 2020

This section is to be taken home, completed, and turned in through Canvas by 10:10 Thursday Dec 17th. There is no time limit and you do not need to type up your solutions to get full marks although the answers should use correct symbols and be well edited.

You may discuss this problem with anyone else from the class and use the class resources posted on Canvas. You may not consult anyone or any resource that is not affiliated with the class such as tutors, websites, or other textbooks. Please follow these rules, I am trusting you to be ethical.

1. Clearly define a set S along with appropriate binary operator(s) that satisfy the following. Briefly justify how each condition is met.
   - [1] The set S has more than 5 elements,
   - [1] forms a non-Abelian group with a binary operator (we will denote with $+$),
   - [1] has at least one subgroup of order $\frac{1}{2}|S|$.
2. Clearly define a set T along with appropriate binary operator(s) that satisfy the following. Briefly justify how each condition is met.
   - [1] The set T has more than 5 elements,
   - [1] forms a ring with a binary operators (we will denote them with + and *), and
   - [1] there exist non-zero elements s and r in S such that $r * s = 0$.
3. Consider Dr. Vanderpool's proof written for the theorem below.
   - [2] Identify any logical errors. (There is more than one!!!)
   - [2] Provide concrete suggestions for how the proof can be improved.

   Theorem 1. *Let G and H be groups and let $\phi : G \to H$ be a homomorphism, then* $\ker(\phi)$ *is a subgroup of H.*

   Proof. We will use the Two-Step Theorem (2.6). That is, we will thus verify if $h \in H$ then $h^{-1} \in H$ and that H is closed.

   First begin with h. Since G is a group, we know that inverses exist, thus we have g^{-1} such that $hg^{-1} = e$. We thus have inverses.

   Secondly we check that $\ker(\phi)$ is closed. Let $a, b \in \ker(\phi)$. We need to show that $ab \in \ker(\phi)$. Note that $\phi(a) = 0$ and $\phi(b) = 0$ because $a, b \in \ker(\phi)$ and by definition of kernel. Since ϕ is a homomorphism we can perform the following conputation:

   $$\phi(ab) = \phi(a)\phi(b) = ee = e$$

   .

   Thus $ab \in \ker(\phi)$ which is what we wanted to show.

   □