Key

Vanderpool **TMath 402** Autumn 2020

True/False: If the statement is false, give a counterexample.
If the statement is *always* true, give a brief explanation of why it is (not *just* an example!).

1. [3] (§2.6) Let G_1 and G_2 be groups such that $G_1 \cong G_2$. Then the Caley Diagram of G_1 matches the Caley Diagram of G_2.

start (+.5)
know Cayley Diag (+1)

false. (+.5) There are many Caley Diagrams for each group. If $G_1 \cong G_2$ then there exists matching Caley Diagrams, but arbitrary Caley Diagrams may not match.

(+1) Ex: Let $G_1 = \mathbb{Z} = G_2$.

Caley Diagram for $G_1 = \langle 1 \rangle$

$-1 \to 0 \to 1 \to 2 \to 3$

Caley Diagram for $G_2 = \langle 2, 3 \rangle$

$-2 \to 0 \to 2 \to 4 \to 6$

$\to -1 \to 1 \to 3 \to 5 \to 7$

2. [3] (§3.3) Consider the integers $\mathbb{Z}$ under addition. The map $\theta : \mathbb{Z} \to \mathbb{Z}$ defined by $\theta(x) = 9x$ satisfies the homomorphic property.

start (+.5)

True (+.5) Let $x, y \in \mathbb{Z}$. We verify $\theta(x+y) = \theta(x) + \theta(y)$ (+1)

Computing left hand side: $\theta(x+y) = 9(x+y)$

$= 9x + 9y$ b/c distribution in $\mathbb{Z}$

$= \theta(x) + \theta(y)$

$=$ right hand side

proved/showed (+1)

3. [3] (§3.1) If H is an abelian subgroup of G, then G is abelian.

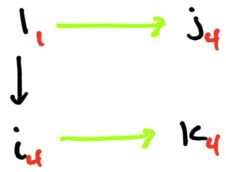
start (+.5)

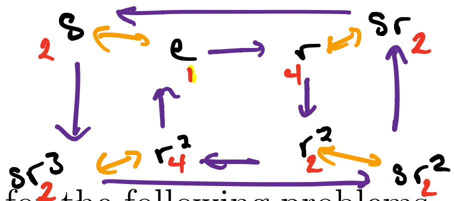
False (+.5) Let $H = \{e\}$ and $G = D_4$.
Note D_4 is not abelian ($rs \neq sr$) even though H is abelian.

ex (+1)

abelian def (+.5)
subgroup def (+.5)

orders





Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

4. [6] (Chapter 2) Identify three non-isomorphic groups of order 8 (you can use the same notation for the groups we used in class). You must explain why each group is not isomorphic to the others.

(+3) lists groups of order 8

$\mathbb{Z}_8$, D_4 , Q (quaternions)

$\mathbb{Z}_8$ is abelian whereas the other two are not. (+2)

(+1)

Notice that there are only two elements in D_4 with order 4 (r and r^3).
However Q has at least three elements with order 4 (i, j & k)
so Q has more elements of order 4 than D_4.

5. (§2.4 & 8.1) For each question below, find a set and a binary operator(s) that satisfy the criteria, if one exists. If no set and binary operator(s) exist, explain why.

set (+.5)
operator (+.5)
cyclic (+.5)
non abelian (+.5)

(a) [3] a cyclic, but non-abelian group

Not possible. If G is cyclic, then $\exists\, g \in G \ni G = \langle g \rangle$.

explain (+1)

Then $\forall x, y \in G$, $\exists\, i, j \in \mathbb{Z} \ni x = g^i$ and $y = g^j$.
Then $xy = g^i g^j = g^{i+j} = g^{j+i}$ b/c $i, j \in \mathbb{Z}$
$= g^j g^i = yx$.

start (+.5)
set (+.5)
operators (+1)
a ring (+.5)
not commutative (+.5)

(b) [3] a ring, but not commutative

$M_{2\times 2}(\mathbb{R})$ with matrix addition & matrix multiplication

recall matrix mult. does not commute, eg

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 8 & -2 \end{bmatrix} \quad \text{but}$$

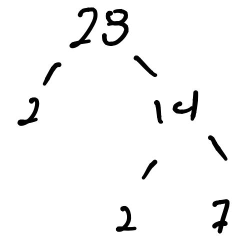
$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix}$$

start (+.5)
set (+.5)
operators (+1)
integral domain (+.5)
not a field (+.5)

(c) [3] an integral domain, but not a field

$(\mathbb{Z}, +, \times)$

note that there are no zero divisors in $\mathbb{Z}$ but integers larger than 1 (ex 3) have no multiplicative inverse.



6. Let $G = \langle a \rangle$ be a group generated by a group of order 28.

(a) [2] (§4.1) Find all the elements that generate G.

(+1) i.e. all powers of a that are relatively prime

Powers of a (+.5)

(+.5) $\{ a^1, a^3, a^5, a^9, a^{11}, a^{13}, a^{15}, {}^{17}, a^{19}, a^{23}, a^{25}, a^{27}$

(b) [2] (§4.1) Find an element in G that has order 4.

start (+.5)

def of order/thm (+1)

got one (+.5)

Want $a^m \in G \ni 4 = \frac{28}{\gcd(28, m)} \Rightarrow \gcd(28,m) = 7$

a^7 or a^{21} would work

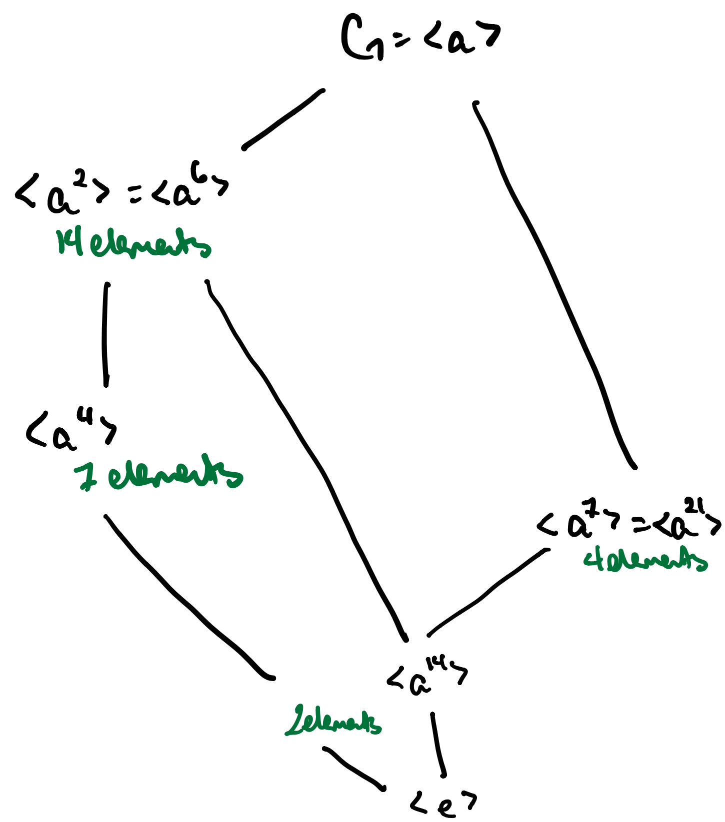
(c) [4] (§3.2 & 4.1) Draw the subgroup lattice for G.

found all the subgroups (+1)

levels/placement (+1)

trivial & non proper subgps (+1)

Containment (+1)



Using the same rubric as on the written HW

7. [8] Choose *ONE* of the following theorems to prove. Clearly identify which of the two you are proving and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

Theorem 1. *Let $\phi : G_1 \to G_2$ be a function between two groups that satisfies the homomorphic property. The kernel of ϕ is defined:*

$$K_\phi = \{g \in G_1 | \phi(g) = e_2\}$$

where e_1 and e_2 are the identities of G_1 and G_2 respectively. Prove that $K_\phi = \{e_1\}$ if and only if ϕ is one-to-one.

Theorem 2. *Let G be a group with identity element e. If $x^2 = e$ for all $x \in G$, then G is abelian.*

Proof of Thm 1: We show each direction seperately.
$\Rightarrow$ Assume $K_\phi = \{e_1\}$, we want to show ϕ is one-to-one.
Let $x, y \in G_1$ such that $\phi(x) = \phi(y)$. To show ϕ is one-to-one we will show that $x = y$.
Since $\phi(x) = \phi(y)$, ϕ has the homomorphic property, and that ϕ preserves inverses,
$\phi(x)\phi(y)^{-1} = \phi(y)\phi(y)^{-1}$ or $\phi(xy^{-1}) = e_2$
Thus $xy^{-1} \in Ker_\phi$
Since $Ker_\phi = \{e_1\}$ we know $xy^{-1} = e_1$ thus $x = y$.
$\Leftarrow$ Assume ϕ is one-to-one, we will show $Ker_\phi = \{e_1\}$.
Note the homomorphic property $\Rightarrow e_1 \in Ker_\phi$, thus we need only show e_1 is the only element in Ker_ϕ.
Let $x \in Ker_\phi$, then $\phi(x) = e_2 \Rightarrow \phi(x) = \phi(e_1)$. Since ϕ is one-to-one, $x = e_1$. Thus $Ker_\phi = \{e_1\}$ //

OR

Pf of Thm 2: We assume $\forall x \in G$, then $x^2 = e$ and will show G is abelian.
That is, we will show $\forall g, h \in G$ that $gh = hg$.
Consider $(gh)^2$. Note $gh \in G$, so our assumption $\Rightarrow (gh)^2 = e$.
Thus $ghgh = e$.
Apply h^{-1} on the right of each side to obtain $ghg = h^{-1}$.
Apply g^{-1} on the right of each side to obtain $gh = h^{-1}g^{-1}$. $*$
Since $h^2 = e$, we know $h^{-1} = h$. Similarly $g^{-1} = g$.
Thus equation $*$ can be simplified to $gh = hg$ which was what we wanted. //

Take Home Final

TMath 402

Autumn 2020

This section is to be taken home, completed, and turned in through Canvas by 10:10 Thursday Dec 17th. There is no time limit and you do not need to type up your solutions to get full marks although the answers should use correct symbols and be well edited.

You may discuss this problem with anyone else from the class and use the class resources posted on Canvas. You may not consult anyone or any resource that is not affiliated with the class such as tutors, websites, or other textbooks. Please follow these rules, I am trusting you to be ethical.

1. Clearly define a set S along with appropriate binary operator(s) that satisfy the following. Briefly justify how each condition is met.
   - [1] The set S has more than 5 elements,
   - [1] forms a non-Abelian group with a binary operator (we will denote with $+$),
   - [1] has at least one subgroup of order $\frac{1}{2}|S|$.
2. Clearly define a set T along with appropriate binary operator(s) that satisfy the following. Briefly justify how each condition is met.
   - [1] The set T has more than 5 elements,
   - [1] forms a ring with a binary operators (we will denote them with $+$ and $*$), and
   - [1] there exist non-zero elements s and r in S such that $r * s = 0$.
3. Consider Dr. Vanderpool's proof written for the theorem below.
   - [2] Identify any logical errors. (There is more than one!!!)
   - [2] Provide concrete suggestions for how the proof can be improved.

Theorem 1. *Let G and H be groups and let $\phi : G \to H$ be a homomorphism, then* $\ker(\phi)$ *is a subgroup of* $\cancel{H}$.

should be in G (+.5)

Proof. We will use the Two-Step Theorem (2.6). That is, we will thus verify if $h \in H$ then $h^{-1} \in H$ and that H is closed. Need to also check nonempty

where does h live? (+1)

indicate what you are assuming and what you need to show.

First begin with h. Since G is a group, we know that inverses exist, thus we have g^{-1} such that $hg^{-1} = e$. We thus have inverses. Need inverse to exist in Ker ϕ (+.5)

Secondly we check that $\ker(\phi)$ is closed. Let $a, b \in \ker(\phi)$. We need to show that $ab \in \ker(\phi)$. Note that $\phi(a) = 0$ and $\phi(b) = 0$ because $a, b \in \ker(\phi)$ and by definition of kernel. Since ϕ is a homomorphism we can perform the following conputation:

$$\phi(ab) = \phi(a)\phi(b) = ee = e$$

.

call identity 0 but then e. Keep notation consistent?

Thus $ab \in \ker(\phi)$ which is what we wanted to show. ∎

all errors are in fact errors (+.5)
find an error (+.5)

start (+.5)
make a reasonable suggestion (+.5)