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Exam 3 TMath 344 Spring 2019

Key

Show all your work. Reasonable supporting work must be shown to earn credit.

1. [8] Let A and B be points on the half plane model of hyperbolic geometry. Let ϕ be a fold. Determine if the following make sense/*could* be true and be sure to justify your answer:

   (a) $A = B$ Sense (+1) {both A + B are points. A and B could be the same point. (+1)

   (b) $B = (3, e^t)$ where $-\infty < t < \infty$ No sense. (+1) B is a Point but (+1) [$(3, e^t)$ as t varies is a line (lots of points)]

   (c) $A = \ln(6)$ No sense, (+1) A is a point in the half plane so it should have an x & y coordinate. (+1)

   (d) $\phi(A) = A$ Sense. (+1) A fold ø can send a point back to itself (if A is on the crease) (+1)

2. [4] (§7.1) Critique the following theorem and proof.

Theorem 1. *Let $l \parallel m$ and ϕ a fold so that $\phi(l) = m$. Let n be the crease of ϕ. Then $l \parallel n \parallel m$.*

logic error (+2)
make comment (+1)
sense/true (+1)

Proof. Assume not, and let $P = l \cap n$. Then $\phi(P) = P \in l$ since $P \in n$. Thus $\phi(P) \notin m$, a contradiction. This completes the proof. □

Contradiction would imply two cases:
1) $n \nparallel l$ or 2) $n \nparallel m$

The above proof considers only case 1 (although a WLOG could be used to fix this?)

Note the contradiction ($\phi(P) \notin m$) only works if l & m are distinct lines.

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★ fix

Theorem Playfaire's. *Given a line l, and a point $A \notin l$, there is a unique line m through A and parallel to l.*

Recall a postulate is an assumed fact. Recall also that the parallel postulate (also known as Euclid's 5th postulate) is the statement "If two parallel lines are cut by a transversal, then alternate interior angles are congruent." Since mathematicians spent a lot of time trying to prove the parallel postulate followed from the previous postulates, there are many statements that are equivalent to the parallel postulate. For example, Playfaire's Theorem is true if and only if the parallel postulate is true.

stat (1.5)
sense (1.5)
got one (+1)

3. [2] Provide another statement that is equivalent to the parallel postulate (that is not Playfaire's Theorem). If 2 // lines are cut by a transversal then the alt. ext. angles are cong.

The sum of angles in a triangle add up to 180° or π radians.
∃ a pair of straight lines that are a constant distance from each other.
There ∃ a Δ whose angles add up to 180° or π radians.
The sum of angles is the same for every triangle.

★

4. Consider how Playfaire's Theorem (and thus the parallel postulate) could *not* be true and examine the ramifications. In particular, describe ~~the consequences~~ what geometry is created if:

(a) 2 there is no line through A and parallel to l.

If ∄ thru A and // to l we arrive at spherical geometry. (+1)

stat (1.5)
sense (1.5)

(b) 2 there are 2 distinct lines through A & parallel to l.

If ∃ two distinct lines thru A + // to l we arrive at

(+1) [hyperbolic geometry.

Note, the statement does not say "exactly 2 distinct lines" so there may be more? (see part c)

stat (1.5)
sense (1.5)

(c) 1 there are 3 distinct lines through A & parallel to l.

The result of assuming there are 3 distinct lines is no diff. than assuming there are 2 in that we arrive again at hyperbolic geometry. (+1)

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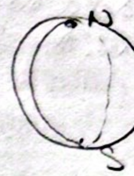
fails this first Postulate and how the Postulate

5. [~~8~~ 4] (2Dpostulate Activity) Spherical geometry fails to satisfy the first Postulate ("Any two points determine a unique line."). Explain how the postulate can be modified to resolve this problem.

(+.5) each

what S^2 +.5 (+2) Fails b/c anti-podal points define an infinite # of lines between them
ex The North + South Poles are connected by any
great circle connecting them



(+1.5) Fix: Any two non-antipodal points determine a ! line
(insert this)

6. Consider the unit sphere in $\mathbb{R}^3$ model for spherical geometry. Let $A = (\frac{1}{3}, \frac{2}{3}, \frac{-2}{3})$ and $B = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

(a) [1] Find the coordinates for $\dot{A}$

$$\left(-\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$$

(b) [3] (§8.1 #7) Find an equation for $\overleftrightarrow{AB}$

Need to find the plane thru A, B + $(0,0,0)$ so a, b & c ∃ (+1)

$$\begin{cases} a\frac{1}{3} + b\frac{2}{3} + c\frac{-2}{3} = 0 \\ a\frac{2}{3} + b\frac{2}{3} + c\frac{1}{3} = 0 \end{cases}$$

System of equations ?

alg (+1)

$$\left[\begin{array}{ccc|c} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} & 0 \end{array}\right] \xrightarrow[3R_2]{3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right]$$

$$R_1 + R_2 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right] \quad \text{so} \quad \begin{array}{l} a + 3c = 0 \\ -2b + 5c = 0 \end{array}$$

Let $c = 1 \Rightarrow a = -3 \quad b = 5/2$ so

$-3x + \frac{5}{2}y + z = 0$ works and $-6x + 5y + 2z = 0$

(+.5) equation of a plane
(+.5) understand parameter/notation

— OR —

(+.5) use Thm

(+1.5) $\left[\frac{2}{3}\cdot\frac{1}{3} - \frac{2}{3}\left(\frac{-2}{3}\right)\right]x + \left[\frac{2}{3}\left(\frac{-2}{3}\right) - \frac{1}{3}\cdot\frac{1}{3}\right]y + \left[\frac{1}{3}\cdot\frac{2}{3} - \frac{2}{3}\cdot\frac{2}{3}\right]z = 0$

$\left[\frac{2}{9} + \frac{4}{9}\right]x + \left[\frac{-4}{9} - \frac{1}{9}\right]y + \left[\frac{2}{9} - \frac{4}{9}\right]z = 0$

(+.5) $\frac{2}{3}x - \frac{5}{9}y - \frac{2}{9}z = 0$ (+.5) eq of plane

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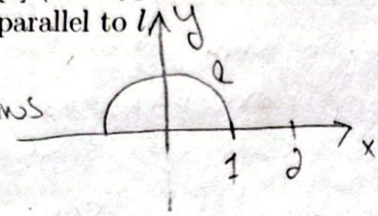
7. Consider the line l in the half plane model of hyperbolic geometry defined by $x^2+y^2=1$

(+.5) start
(+1) got 1 line // to l
(+1) knows def of lines
(+.5) got more than 2 lines

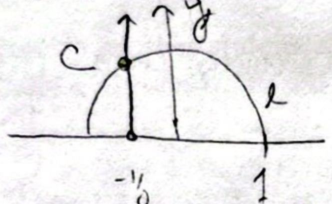
(a) [3] (2D HyperbolicActivity #1) Find a family (more than two!) of lines that are parallel to l



how about
$\{(x,y) \mid x^2+y^2=r^2 \text{ where } r>1\}$
so circles centered at $(0,0)$ w/ larger radia
-or-
$\{(v,y) \mid v<-1 \text{ or } 1<v\}$ so vert. semi lines.

(b) [3] (§7.4 #27) Let $A = (\frac{-1}{2}, \frac{7}{25})$ and $B = (\frac{-1}{2}, \frac{10}{3})$. Find the point $C = l \cap \overleftrightarrow{AB}$.

(+.5) start
(+.5) knows def of lines



$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$

want to find $(x,y) \ni x^2+y^2=1$ and $x=-\frac{1}{2}$ (+1)

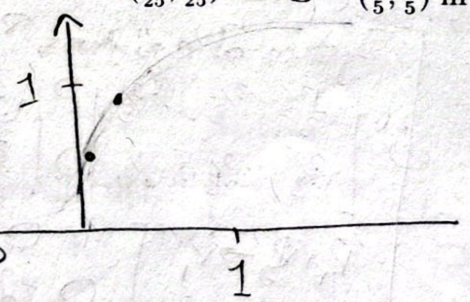
So $(-\frac{1}{2})^2+y^2=1 \Rightarrow y^2 = 1-\frac{1}{4} = \frac{3}{4}$ (+1)
$\Rightarrow y = \sqrt{3/4}$ or $\frac{\sqrt{3}}{2}$

(do dont have to $l=r$ next line?)

8. [7] (5/28 Lecture) Find the distance between points $A = (\frac{1}{25}, \frac{7}{25})$ and $B = (\frac{2}{5}, \frac{4}{5})$ in the half plane model of hyperbolic geometry.

start (+.5)

(+2) Note A & B are not on the same vertical line so we are looking for h & r ∋



(+.5) $[(x-h)^2+y^2=r^2$ given the pts A & B

(+2) finding h
$\left[\begin{array}{l}(\frac{1}{25}-h)^2+(\frac{7}{25})^2=r^2 \\ (\frac{2}{5}-h)^2+(\frac{4}{5})^2=r^2\end{array}\right\} \rightarrow (\frac{1}{25}-h)^2+(\frac{7}{25})^2-(\frac{2}{5}-h)^2-(\frac{4}{5})^2=0$

$\Rightarrow \frac{1}{625}-\frac{2h}{25}+h^2+\frac{49}{625}-(\frac{4}{25}-\frac{4h}{5}+h^2)-\frac{16}{25}=0$

$\Rightarrow \underbrace{\frac{2}{25}}-\frac{2h}{25}+\cancel{h^2}-\frac{4}{25}+\frac{4h}{5}-\cancel{h^2}-\frac{16}{25}=0 \Rightarrow h=1$

find r: $[(\frac{2}{5}-1)^2+(\frac{4}{5})^2=r^2 \Rightarrow \frac{9}{25}+\frac{16}{25}=r^2 \Rightarrow r=1$

(+1) So $(1+\tanh(t), \operatorname{sech}(t))$ where $-\infty<t<\infty$, is the line $\overleftrightarrow{AB}$

(+2) [@ point A when $1+\tanh(t)=\frac{1}{25} \Rightarrow t=\operatorname{arctanh}(\frac{1}{25}-1)=\operatorname{arctanh}(-\frac{24}{25})$
@ point B when $1+\tanh(t)=\frac{2}{5} \Rightarrow t=\operatorname{arctanh}(\frac{2}{5}-1)=\operatorname{arctanh}(-\frac{3}{5})$

(+1) [So distance is $|\operatorname{arctanh}(\frac{-24}{25})-\operatorname{arctanh}(\frac{-3}{5})|$

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Recall lines on the unit sphere model of spherical geometry are intersections of the sphere (defined by $x^2 + y^2 + z^2 = 1$) with planes in $\mathbb{R}^3$ through the origin (defined by $ax + by + cz = 0$ where $(a, b, c) \neq (0, 0, 0)$). For each line in the unit sphere there is thus exactly two (antipodal) points $\pm(a, b, c)$ in the sphere whose coordinates are coefficients for the line.
↳ the vectors normal to the plane

Let us create a new geometry model called Projective Geometry and define "point" to mean pairs of antipodal points in the unit sphere. Define "line" as the set of points that are both on the sphere and on a plane in $\mathbb{R}^3$ that passes through the origin.

9. [2] Does the Projective Geometry satisfy Postulate 1? Justify your answer.

(.5) [yes?

We can use the same (and same plane) reasoning used for the unit sphere model of spherical geometry.

use pt of line (.5)
justification (1)

Note however we no longer need the condition that our points are not antipodal because of the definition of point?

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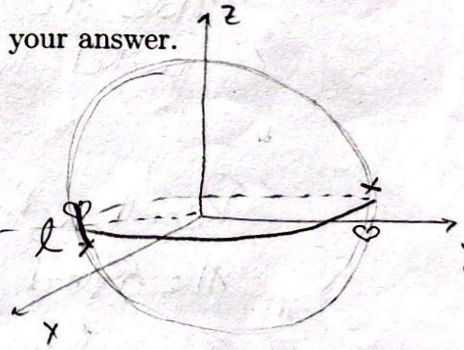
10. [2] Does the Projective Geometry satisfy Postulate 3? Justify your answer.



Consider the line ℓ defined by $z=0$ (so the equator).

use pt of line (.5)
justification (1)

Notice the top hemisphere is identified with the bottom hemisphere since antipodal points are one 'point'.

Thus we have Projective Geometry decomposed to:

ℓ : $\{(x,y,z) \mid z=0$ and $x^2+y^2=1\}$ ie the equator

H_+ : $\{(x,y,z) \mid z>0$ and $x^2+y^2+z^2= \}$ ie upper hemisphere

H_- : $\emptyset$

Consider the points where $x=0$ and z is slightly positive. Marked with an x and ♡ above. The line connecting x to ♡ crosses the line ℓ ⇒ x and ♡ are not on the same side of ℓ but they should be by post 3.

(.5) [Fails Postulate 3.

Note: all lines intersect each other so Post 3 will always fail

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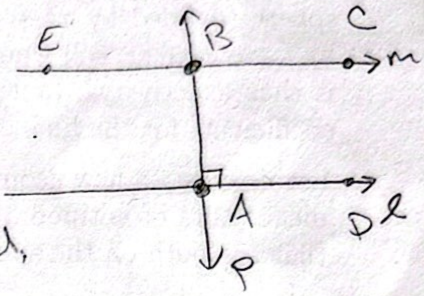
11. [6] Let l and m be two lines in hyperbolic space with a common perpendicular. Prove that l and m are ultra parallel.



start (+.5)
notation/def (+1)
style (+.5)
logic (+2)
edited (+1)
hyperbolic results +still (+1)

We first show that l and m are parallel and then show m is not horo-parallel to l to imply l and m are ultraparallel.

Let p be the common perpendicular to l and m. Label the intersection by points as done in the figure above.

Notice since $p \perp l$, $\angle DAB = \pi/2$ and since $p \perp m$, $\angle ABE = \pi/2$.

So the alternating interior angles are congruent $\Rightarrow m \parallel l$ (Thm 6.1)

[note the $\parallel$ postulate works the other way that is parallel lines imply congruent alt. int. angles]

Recall a Cor. proved in class, (and then in the HW?), that the angle of parallelism is acute or less than $\pi/2$.

Recall the angle of parallelism for l would be the angle created by p and a line horo-parallel to l. Thus the line horoparallel to l is not m (which has an angle of $\pi/2$) $\Rightarrow m$ is ultraparallel to l.

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