Topology & Maps

While working in a group make sure you:

- Expect to make mistakes but be sure to reflect/learn from them!
- Are civil and are aware of your impact on others.
- Assume and engage with the strongest argument while assuming best intent.

Definition 0.1. Let X be a set. A *topology on* X is a collection sets, τ (which is a subset of the powerset of X) that define the *open sets* of X. The collection τ satisfies:

1. $X \in \tau$ and $\emptyset \in \tau$,
2. if $U_i \in \tau$ for all i in some index set A, then $\left(\bigcup_{i \in A} U_i\right) \in \tau$, and
3. if $U_i \in \tau$ for all $i \in A$, and $|A| < \infty$, then $\left(\bigcap_{i \in A} U_i\right) \in \tau$.

1. Consider the following X and τ. Determine if they define a topology.

   (a) Let $X = \{v_1, v_2, v_3\}$ and $\tau_0 = \{\{v_1, v_2, v_3\}, \emptyset\}$

   (b) Let $X = \{v_1, v_2, v_3\}$ and $\tau_1 = \{\{v_1, v_2, v_3\}, \{v_1\}, \{v_1, v_2\}, \emptyset\}$

   (c) Let $X = \{v_1, v_2, v_3\}$ and $\tau_2 = \{\{v_1, v_2, v_3\}, \{v_1\}, \{v_2\}, \emptyset\}$

2. Create a topology on the set $X = \{v_1, v_2, v_3\}$ so that we can 'separate' v_1 from v_2. That is, we can find two open sets U_1 and U_2 so that $v_1 \in U_1$, $v_2 \in U_2$ but $U_1 \cap U_2 = \emptyset$.

Definition 0.2. Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if for every open set $U \subseteq Y$, the set $f^{-1}(U) \subseteq X$ is open. We also call continuous functions *maps*.

3. Let $X = \{v_1, v_2, v_3\}$ and $f : X \to X$ where $f(v_i) = v_i$ for $i \in \{1, 2, 3\}$. Does f define a continuous map if:

   (a) the domain of f has topology τ_0 and the range has topology τ_1?

   (b) the domain of f has topology τ_1 and the range has topology τ_0?

4. Consider the continuous map $f : \mathbb{R} \to \mathbb{R}$ (where $\mathbb{R}$ has the usual topology) defined by $f(x) = (x+3)(x+1)(x-2)(x-4)$. "Approximate" $f^{-1}(\mathcal{B}_{.01}(0))$ by determining how many disjoint open sets there are.