TMATH 342: Example Proof

Theorem 0.1. *A set isomorphism is an equivalence relation on the collection of all graphs.*

Proof. Recall an equivalence relation $\sim$ must be reflexive, symmetric, and transitive. We thus verify these three properties to show set isomorphisms satisfy the requirements of an equivalence relation on sets.

Let A, B and C be sets. Define the equivalence relation $\sim$ so that $A \sim B$ if there exists a set isomorphism between A and B. Recall the definition of a set isomorphism, that there is a $f : A \to B$ with a two-sided inverse $g : B \to A$ so $f \circ g = id_B$ and $g \circ f = id_A$.

We first verify $\sim$ is reflexive. Assume that $A \sim A$, we need to show that $A \sim A$. Note that we can write down the isomorphism here as $f : A \to A$ by $f(a) = a$ for all $a \in A$. The inverse g, is in fact the same identity map, thus we have a $A \sim A$.

To check $\sim$ is symmetric we assume $A \sim B$ and want to show $B \sim A$. Notice since $A \sim B$ we know there exists and isomorphism $f : A \to B$ that has a two-sided inverse $g : B \to A$. Thus we have an isomorphism $g : B \to A$ with a two-sided inverse $f : A \to B$ meaning that $B \sim A$.

We last check $\sim$ is transitive. Assume that $A \sim B$ and $B \sim C$. We need to show that $A \sim C$. Since we know $A \sim B$ we have an isomorphism $f : A \to B$ with a two-sided inverse $g : B \to A$. So $f \circ g = id_B$ and $g \circ f = id_A$. Since we know that $B \sim C$ we have an isomorphism $j : B \to C$ with a two-sided inverse $k : C \to B$. So $j \circ k = id_C$ and $k \circ j = id_B$.

Consider the compositions, or $j \circ f : A \to C$ and $g \circ k : C \to A$. Notice since map composition is associative:

$$(j \circ f) \circ (g \circ k) = j \circ (f \circ g) \circ k = j \circ id_B \circ k = j \circ k = id_C$$

$$(g \circ k) \circ (j \circ f) = g \circ (k \circ j) \circ f = g \circ id_B \circ f = g \circ f = id_A$$

We thus have an isomorphism $j \circ f : A \to C$ with a two-sided inverse meaning that $A \sim C$ which completes the proof. □