Math 341: Quiz 5

Name: KEY

True/False: If the statement is true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. [3] A list of vectors $\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}$ with repeated vectors is linearly dependent.

true (1)
started (1)
got it (1)

True Let $\vec{v_i}, \vec{v_j} \in \{\vec{v_1}, \ldots \vec{v_n}\}$ be ~~the~~ the repeated vector. That is $i \neq j$, but $\vec{v_i} = \vec{v_j}$.

Then $0\vec{v_1} + \ldots + 0\vec{v_{i-1}} + \vec{v_i} + 0\vec{v_{i+1}} + \ldots + 0\vec{v_{j-1}} - \vec{v_j} + 0\vec{v_{j+1}} + \ldots + 0\vec{v_n}$ equals $\vec{0}$ but not all the scalars are zero.

Thus $\{\vec{v_1}, \ldots \vec{v_n}\}$ is lin. dep.

2. [3] The set $\{1-x, x-x^2, 1-x\}$ is linearly independent in the vector space of all polynomials of degree 2 or less.

false (1)
looked for counter (1)
got it (1)

False See above.

-or-

$(1-x) + 0(x-x^2) - (1-x) = \vec{0}$

but not all the scalars are zero

3. [3] If $V = \text{span}\{\vec{v_1}, \ldots \vec{v_n}\}$, then $\{\vec{v_1}, \ldots \vec{v_n}\}$ is a basis of V.

False $\mathbb{R}^2 = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

but $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is lin. dep

and $\therefore$ not a basis.

4. [3] Linear operators preserve linear independence. That is given a linear operator $T : V \to W$ and a linearly independent set $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\} \in V$, the set $\{T(\vec{v_1}), T(\vec{v_2}), \ldots T(\vec{v_n})\}$ is linearly independent.

False Let $T: \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

note $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \subset \mathbb{R}^2$ is lin indep. but

$\left\{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right\} = \left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ is lin dep.

5. Let $V = \{\vec{T} : \mathbb{R}^3 \to \mathbb{R}^2 | \vec{T} \text{ is a linear function}\}$. The elements in our set are thus *linear maps.* Define vector addition and scalar multiplication by the following:

$$(\vec{T}+\vec{R})\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \vec{T}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + \vec{R}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \qquad (\alpha\vec{T})\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \alpha\left(\vec{T}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$$

where $\alpha \in \mathbb{R}$, and $\vec{T}$, $\vec{R} \in V$.

It is given that V forms a vector space under the defined operations above. In particular note that the zero vector is the linear map defined by $\vec{0}([x\ y\ z]^\top) = [0\ 0]^\top$ for all $[x\ y\ z]^\top \in \mathbb{R}^3$.

Recall on last weeks quiz that you showed

$$S = \{\vec{T} \in V | \exists\ A \in \mathrm{Mat}_{2x3}(\mathbb{R}) \text{ so that } \vec{T}([x\ y\ z]^\top) = A[x\ y\ z]^\top \text{ for all } [x\ y\ z]^\top \in \mathbb{R}^3\}$$

forms a subspace of V. We will now show that $V \subset S$ and thus V actually *equals* S.

You have two options:

(a) Show that $V \subset S$. That is, given an arbitrary linear operator $T : \mathbb{R}^3 \to \mathbb{R}^2$, construct a 2x3 matrix A that has the property that $\vec{T}([x\ y\ z]^\top) = A[x\ y\ z]^\top$ for *any* vector $[x\ y\ z]^\top$ in $\mathbb{R}^3$.

(b) Follow the steps on the next page that will walk you through the proof.

(a) [4] The trick is to make use of a basis of $\mathbb{R}^3$.

Let $\{e_1, e_2, e_3\}$ be a standard basis for $\mathbb{R}^3$ and T be an arbitrary linear operator in V. Let A_1 be the 2x1 vector in $\mathbb{R}^2$ that results from T acting on e_1, that is $A_1 = T(e_1)$. Similarly let $A_2 = T(e_2)$ and $A_3 = T(e_3)$. Define $A = [A_1\ A_2\ A_3]$. Verify that T and A send the standard basis to the same element in $\mathbb{R}^2$. (ie. Verify that $T(e_i) = Ae_i$ for i=1,2 or 3 by making use of matrix multiplication.)

Stated (+1)
Showed understanding (+1)
right idea (+1)

$$Ae_1 = [A_1\ A_2\ A_3]\begin{bmatrix}1\\0\\0\end{bmatrix} = 1A_1 + 0A_2 + 0A_3 = A_1 = T(e_1)$$

$$Ae_2 = [A_1\ A_2\ A_3]\begin{bmatrix}0\\1\\0\end{bmatrix} = 0A_1 + 1A_2 + 0A_3 = A_2 = T(e_2)$$

$$Ae_3 = [A_1\ A_2\ A_3]\begin{bmatrix}0\\0\\1\end{bmatrix} = 0A_1 + 0A_2 + 1A_3 = A_3 = T(e_3)$$

(b) [4] Let $[x\ y\ z]^T \in \mathbb{R}^3$. Since $\{e_1, e_2, e_3\}$ is a spanning set, we can write $[x\ y\ z]^T$as $c_1e_1 + c_2e_2 + c_3e_3$. Use linearity, properties of matrix multiplication, and part a) to show that $A[x\ y\ z]^T = T([x\ y\ z]^T)$.

Stated (+1)
Showed understanding (+1)
right idea (+1)

$$A\begin{bmatrix}x\\y\\z\end{bmatrix} = A(c_1e_1 + c_2e_2 + c_3e_3) =$$ by prop of matrix mult.

$$= c_1Ae_1 + c_2Ae_2 + c_3Ae_3$$ by part a

$$= c_1T(e_1) + c_2T(e_2) + c_3T(e_3)$$ since T is linear

$$= T(c_1e_1 + c_2e_2 + c_3e_3)$$ by choice of c_1, c_2, c_3

$$= T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$$