Math 341: Quiz 5

Name:

True/False: If the statement is true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. [3] A list of vectors $\vec{v_1}, \vec{v_2}, ...\vec{v_n}$ with repeated vectors is linearly dependent.

2. [3] The set $\{1 - x, x - x^2, 1 - x\}$ is linearly independent in the vector space of all polynomials of degree 2 or less.

3. [3] If $V = \text{span}\{\vec{v_1}, ...\vec{v_n}\}$, then $\{\vec{v_1}, ...\vec{v_n}\}$ is a basis of V.

4. [3] Linear operators preserve linear independence. That is given a linear operator $T : V \to W$ and a linearly independent set $\{\vec{v_1}, \vec{v_2}, ...\vec{v_n}\} \in V$, the set $\{T(\vec{v_1}), T(\vec{v_2}), ...T(\vec{v_n})\}$ is linearly independent.

5. Let $V = \{\vec{T} : \mathbb{R}^3 \to \mathbb{R}^2 | \vec{T} \text{ is a linear function}\}$. The elements in our set are thus *linear maps.* Define vector addition and scalar multiplication by the following:

$$(\vec{T}+\vec{R})\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \vec{T}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + \vec{R}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \qquad (\alpha\vec{T})\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \alpha\left(\vec{T}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$$

where $\alpha \in \mathbb{R}$, and $\vec{T}, \vec{R} \in V$.

It is given that V forms a vector space under the defined operations above. In particular note that the zero vector is the linear map defined by $\vec{0}([x\ y\ z]^\top) = [0\ 0]^\top$ for all $[x\ y\ z]^\top \in \mathbb{R}^3$.

Recall on last weeks quiz that you showed

$$S = \{\vec{T} \in V | \exists\ A \in \text{Mat}_{2x3}(\mathbb{R}) \text{ so that } \vec{T}([x\ y\ z]^\top) = A[x\ y\ z]^\top \text{ for all } [x\ y\ z]^\top \in \mathbb{R}^3\}$$

forms a subspace of V. We will now show that $V \subset S$ and thus V actually *equals* S.

You have two options:

(a) Show that $V \subset S$. That is, given an arbitrary linear operator $T : \mathbb{R}^3 \to \mathbb{R}^2$, construct a 2x3 matrix A that has the property that $\vec{T}([x\ y\ z]^\top) = A[x\ y\ z]^\top$ for *any* vector $[x\ y\ z]^\top$ in $\mathbb{R}^3$.

(b) Follow the steps on the next page that will walk you through the proof.

(a) [4] The trick is to make use of a basis of $\mathbb{R}^3$.

Let $\{e_1, e_2, e_3\}$ be a standard basis for $\mathbb{R}^3$ and T be an arbitrary linear operator in V. Let A_1 be the 2x1 vector in $\mathbb{R}^2$ that results from T acting on e_1, that is $A_1 = T(e_1)$. Similarly let $A_2 = T(e_2)$ and $A_3 = T(e_3)$. Define $A = [A_1\ A_2\ A_3]$. Verify that T and A send the standard basis to the same element in $\mathbb{R}^2$. (ie. Verify that $T(e_i) = Ae_i$ for i=1,2 or 3 by making use of matrix multiplication.)

(b) [4] Let $[x\ y\ z]^T \in \mathbb{R}^3$. Since $\{e_1, e_2, e_3\}$ is a spanning set, we can write $[x\ y\ z]^T$as $c_1e_1 + c_2e_2 + c_3e_3$. Use linearity, properties of matrix multiplication, and part a) to show that $A[x\ y\ z]^T = T([x\ y\ z]^T)$.