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MIDTERM

# Math 341

Summer 2008

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NAME:

KEY

True/False: If the statement is *always* true, give a brief explanation of why it is. If the statement is false, give a counterexample. Let  $A$  and  $B$  be matrices, and  $c$  be a scalar.

1. [4] The matrices  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$  are equal.

(+2) False  $\begin{matrix} \vdots \\ A \end{matrix}$   $\begin{matrix} \vdots \\ B \end{matrix}$

(+2) since  $a_{11} \neq b_{11}$

2. [4] Every matrix can be reduced to one and only one matrix in reduced row form.

(+1) False  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  is reduced and equivalent to  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  which is also reduced  
(+1) looking for  $\times$   
(+2) got it say something false 10

3. [4] If  $cA = 0$ , then either  $c = 0$  or  $A = 0$ .

(+1) True. If  $c=0$  then  $cA = 0 \cdot A = [0a_{ij}] = [0]$   
If  $c \neq 0$  then mult  $cA = 0$  by scalar  $\frac{1}{c}$

(+1) started justifying

(+1) sense

(+1) got it

$\Rightarrow \frac{1}{c}(cA) = \frac{1}{c}0$  by assu  
 $\Rightarrow (\frac{1}{c}c)A = \frac{1}{c}0$  by scalar mult  
 $\Rightarrow A = 0$  so  $a_{ij} = 0 \forall i, j$

4. [4] The matrix  $\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$  is invertible.

(+2) True

justification (+2)  $\det(A) = 2(-2) - 3(1) = -4 - 3 = -7 \neq 0$

5. [4] The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by the rule  $f(x) = x + 1$  is a linear function.

(+2) False. Consider the linear combination  $1+1$

counterex (+2)

$$3 = 2 + 1 = f(2) = f(1+1) \quad \times \quad f(1) + f(1) = (1+1) + (1+1) = 4$$

6. [4] If a matrix  $A$  is diagonal then  $A$  is not upper triangular.

(+2) False

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is both diagonal and upper triangular

(+2)

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

7. [6] Complete the following definition: A map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation if...

given any linear combination  $a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_n\vec{x}_n$  where  $a_i$  are scalars &  $\vec{x}_i$  are  $\mathbb{R}^n$  vectors.

$$T(a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_n\vec{x}_n) = a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \dots + a_nT(\vec{x}_n)$$

8. [10] Let  $A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

- (a) Find the reduced row echelon form of  $A$ .

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 4 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \rightarrow NR_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\ & \xrightarrow{R_3 - R_2 \rightarrow NR_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2 \rightarrow NR_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

- (b) What is the rank and nullity of  $A$ ?

$$\begin{aligned} \text{rank}(A) &= 2 \\ \text{nullity}(A) &= 2 \end{aligned}$$

- (c) Use Gaussian elimination to find the solution set to the following system of linear

$$\text{equations: } \begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 2x_2 + 4x_3 = 8 \\ x_3 = 2 \end{cases}$$

notice  $A$  is the augmented matrix that corresponds to this linear system so we've already done the work

$$\begin{aligned} & \{(x_1, x_2, x_3) \mid x_1 + x_2 = 0, x_3 = 2, x_1 \in \mathbb{R}\} \\ & = \{(x_1, -x_1, 2) \mid x_1 \in \mathbb{R}\} \end{aligned}$$

9. [20] For the following problem let

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Perform the following calculations where possible:

(a)  $3A - B$

$3 \quad - \quad \textcircled{+2} \text{ not possible b/c of sizes}$   
 $(2 \times 2) \quad (2 \times 1)$

(b)  $AB$

$\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \textcircled{+2}$   
 $(2 \times 2) \quad (2 \times 1)$

(c)  $BA$

$(2 \times 1)(2 \times 2) \quad \text{not possible} \textcircled{+2}$

[10] (d) Find the inverse of  $C$  if it exists.

$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \rightarrow NR_2 \\ R_3 - 3R_1 \rightarrow NR_3}} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \end{bmatrix}$   
 $\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \\ 0 & 0 & 4 & -2 & 1 & 0 \end{bmatrix} \xrightarrow{\substack{\text{set up +5} \\ 4R_3}} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix} \xrightarrow{\substack{R_2 + R_3 \rightarrow NR_2 \\ R_1 + R_3 \rightarrow NR_1}} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{4} & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}$

[4] (e) Compute the matrix of minors of  $D$ , ie find  $M(D)$ . ~~do not calculate~~ recognize there is an  $2 \times 2$  of 1's calc errors  $-3$

Find the det of  $C$   
 $\begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} \stackrel{\textcircled{+1}}{=} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{vmatrix} \text{ by above} \quad \text{Check!}$   
 $= -(1)(1)(4)$   
 $= -4 \checkmark$   
 $\begin{bmatrix} 1 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 2 & 0 & 2 & -\frac{1}{2} & \frac{1}{4} & 1 \\ 3 & 1 & 1 & -\frac{1}{2} & \frac{1}{4} & 0 \end{bmatrix}$

expanded down the middle col

$-1 \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -1(2 - (-1)(2)) = -4 \checkmark$

$\frac{3}{2} - 1 - \frac{1}{2}$   
 $\frac{3}{4} - 1 + \frac{1}{4}$

10. [15] Let  $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the linear operator defined by the rule  $T_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  where  $A$  is some matrix and  $A$  acts on the vector by standard matrix multiplication.

(45) (a) What are the dimensions of  $A$ ?

$A$  is a  $2 \times 3$  matrix

(+10) (b) Given that  $T_A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $T_A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ , and  $T_A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ , find the matrix  $A$  that corresponds to the function  $T_A$ .

Let  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

•  $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = T_A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \Rightarrow a_{11} = 0, a_{21} = 0$

•  $\begin{bmatrix} 0 \\ 3 \end{bmatrix} = T_A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \Rightarrow a_{12} = 0, a_{22} = 3$

•  $\begin{bmatrix} 3 \\ 0 \end{bmatrix} = T_A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 3 & a_{23} \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \Rightarrow a_{13} = 3, a_{23} = 0$

Thus  $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$  +1 work.

11. [15] Let  $V$  be all diagonal  $2 \times 2$  matrices with real entries. That is let  $V = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$  with the standard matrix addition and scalar multiplication. Determine if  $V$  is a vector space.

Notice  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$  thus  $V$  is nonempty

Let  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in V$  and  $c, d \in \mathbb{R}$

(1)  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} a+\alpha & 0 \\ 0 & b+\beta \end{bmatrix} \in V$  by the closure of  $+$  in  $\mathbb{R}$

(2)  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} a+\alpha & 0 \\ 0 & b+\beta \end{bmatrix} = \begin{bmatrix} \alpha+a & 0 \\ 0 & \beta+b \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

(3)  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = (\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}) + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$  by assoc. of  $+$  in  $\mathbb{R}$

(4) Consider  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$ . By add. id. from  $\mathbb{R}$ , this works.

(5) Given  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in V$ , we see  $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$  is the additive inv.

(6)  $c \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ca & 0 \\ 0 & cb \end{bmatrix} \in V$  by the closure of mult in  $\mathbb{R}$

(7) (8) By dist. prop of mult over  $+$  in  $\mathbb{R}$  we have

(9)  $(c+d) \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = c \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + d \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$  and

$c(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}) = c \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + c \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$

(10) By assoc. of mult in  $\mathbb{R}$   
 $(cd) \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = c(d \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix})$

Monoid (10) Consider  $1 \in \mathbb{R}$ .  
 The prop. of  $1$  in  $\mathbb{R}$  implies this works

(+2) cite reasons on something  
 (+2) notation  
 (+2) ~~notation~~  
 +1 nothing wrong/added