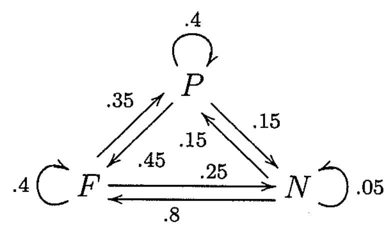
Markov Problems

To create a transition matrix for a Markov process one can have the data handed to them in a list as was done in class or we can make use of graphs to state the data much more concisely.

1. A math teacher from the University of North Texas decided to assign homework based on probabilities. On the first day of class she drew the following graph on the board to tell the students whether to expect a full assignment (F), a partial assignment (P), or no assignment (N) the next day.



The numbers next to the arrows in the above diagram give the probability of moving from the state at the base of the arrow to the state at the tip of the arrow. For instance, if a partial assignment was given on Monday, there would be a 45% chance that a full assignment will be assigned on Tuesday.

(a) The probabilities of having a full, partial, or empty assignment on Monday are x, y and z, respectively. Find the likelihood of each kind of assignment that may be given Tuesday. Write this system down as a product of matrices, indicating which is the transition matrix A.

possibly a full assignment:
$.4x + .45y + .8z$
possibly a partial assg.:
$.35x + .4y + .15z$
possibly no assig.:
$.25x + .15y + .05z$

$$\begin{bmatrix} .4 & .45 & .8 \\ .35 & .4 & .15 \\ .25 & .15 & .05 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) Today is Wednesday and the students have a partial assignment. What is the probability that there will be no homework on Friday?

$$\begin{bmatrix} .4 & .45 & .8 \\ .35 & .4 & .15 \\ .25 & .15 & .05 \end{bmatrix}^2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .48 \\ .34 \\ .18 \end{bmatrix}$$

so 18%

(c) Given that the probabilities of having a full, partial, or empty assignment today are x, y and z, respectively, find the probabilities of each kind of assignment that may be given k class days from now.

$$\begin{bmatrix} .4 & .45 & .8 \\ .35 & .4 & .15 \\ .25 & .15 & .05 \end{bmatrix}^k \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(d) Recall that if there is a vector that all initial values tend toward, then it will be in the null space of $I - A$ where A is still the transition matrix you found in (a). Find $\mathcal{N}(I - A)$.

$$I-A = \begin{bmatrix} .6 & -.45 & -.8 \\ -.35 & .6 & -.15 \\ -.25 & -.15 & .95 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2.\overline{703} \\ 0 & 1 & -1.827 \\ 0 & 0 & 0 \end{bmatrix}$$

rounding error introduced.

Sol set $\{(2.\overline{703}z, 1.827z, z) \mid z \in \mathbb{R}\}$ $= \text{span}\left\{\begin{bmatrix} 2.\overline{703} \\ 1.827 \\ 1 \end{bmatrix}\right\}$

(e) Explain the meaning to the solution you found in (d).

The probabilities of each kind of assignment being given may stabilize in the future to this distribution.

$a \cdot 2.703 + a \cdot 1.827 + a = 1$

$a = .1808$

$\begin{bmatrix} .489 \\ .330 \\ .181 \end{bmatrix} \in \text{span}\left\{\begin{bmatrix} 2.\overline{703} \\ 1.827 \\ 1 \end{bmatrix}\right\}$

↳ vector whose entries all add to 1

We've only shown that this distribution is a steady state - we have not yet shown any initial condition will tend towards it. However experimental evidence gathered from examining the following suggests that this is in fact the case.

$A^{10}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .489 \\ .330 \\ .181 \end{bmatrix}$ $A^{10}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .489 \\ .330 \\ .181 \end{bmatrix}$ $A^{10}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .489 \\ .330 \\ .181 \end{bmatrix}$