in reverse order.

13) We need to show there are more than one $\vec{x}$ that satisfies $A\vec{x} = \vec{b}$.

Since the system is consistent I know there is at least one solution, I'll denote this by $\vec{x}_*$. That is $A\vec{x}_* = \vec{b}$.

Thm 3.15 on pg 200 tells me all the solutions to the system $A\vec{x} = \vec{b}$ take the form

$$\vec{x}_* + \vec{z} \quad \text{where } \vec{z} \in N(A).$$

Thus to show there is more than one solution to $A\vec{x} = \vec{b}$, I need only show there is more than one element in $N(A)$.

I am given that the columns of A contain a redundant vector. Let A_i denote the i^{th} column then the above suggests there exists $c_1 \ldots c_n$ that are not all zero with the property

$$c_1 A_1 + c_2 A_2 + \ldots + c_n A_n = \vec{0}$$

Consider $[c_1, c_2, \ldots c_n]^T$. Note the above implies

$$A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \quad \text{so} \quad [c_1, c_2, \ldots c_n]^T \in N(A)$$

and is not the trivial element. Thus $N(A)$ contains a nontrivial solution.

Recall this implies $\vec{x}_*$ and $\vec{x}_* + [c_1, \ldots c_n]^T$ are both distinct solutions. //

9) $A = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & -1 \\ 3 & 1 & 0 \\ 5 & 2 & -1 \end{bmatrix} \underset{\substack{R_2+3R_1 \\ R_2+5R_1}}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 4 & -3 \\ -2 & 3 \\ 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ -2 & 3 \\ 4 & -3 \end{bmatrix} \underset{\substack{R_2+2R_1 \\ R_3-4R_1}}{\longrightarrow} \begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 0 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

a)

$\dim(\mathcal{C}(A)) = 2$ b/c $\mathcal{C}(A) = \text{span}\left\{ \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

↳ lin. ind. by above.

$\dim(\mathcal{C}(B)) = 2$ b/c $\left\{ \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} \right\}$ are lin. indep by above.

b)

$\dim(U+V)$

$U+V = \text{span}\left\{ \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} \right\}$ need to get a basis out of this.

$= \text{span}\left\{ \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} \right\}$ by work with A