T/F for quiz

find counterex if false

§3.5 # 3, 8, 9, 10, 11, 12, 20, 23

3) $[v_1\ v_2\ w_1] = \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 3 & 2 & 1 \\ 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$

so $2v_2 = w_1$

v_2 can be replaced with w_1 & retain lin. ind. of resulting set

8) Let $\vec{w}_1 = x+1$

note $P_2 = \text{span}\{1, x, x^2\}$

So $P_2 = \text{span}\{x+1, 1, x, x^2\}$

I'll now use S. algorithm to reduce the set to a lin. indep set

$1 \notin \text{span}\{x+1\}$

$x \in \text{span}\{x+1, 1\}$

$x^2 \notin \text{span}\{x+1, 1\}$

So basis: $\{x+1, 1, x^2\}$

9) a) True since $\text{span}\, S \supset V \Rightarrow k \geq n$

since $\text{span}\, S \subset V \Rightarrow k \leq n$ $\therefore k = n$

b) False let $V = \mathbb{R}^2$ $n = 2$

$S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $k = 3$

c) True $\dim(\text{span}\, S) = k$ & since span S is a subspace of V

$k \leq n$

d) True $\dim(\text{span}\, S) = k$ so $k \leq n$ & Cor 3.4

$\Rightarrow \text{span}\, S = V$

e) True if S spans V $\Rightarrow k \geq n$

but $k = n$ so minimal spanning set $\Rightarrow$ S is a basis

f) True Since $\det A \neq 0$ A is invertible

$\Leftrightarrow$ the columns of A are lin. indep

So the first 4 col. carve out a 4 dim. subspace.

10) a) false Let $V = \mathbb{R}^2$ $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

S is lin. ind. but not a basis

b) true the contrapositive was on a quiz

redundant vectors $\Rightarrow$ lin. dep

c) True Since $V = \text{span}\{\vec{v}_2, \vec{v}_3\}$ and $S \subseteq V$

$\exists$ scalars $c_2, c_3 \ni v_1 = c_2\vec{v}_2 + c_3\vec{v}_3 \Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin dep

e) False Consider the set of polynomials.

d) False Given the set, reorder them so you have $\{\vec{0}, \vec{v}_1, \vec{v}_2, .. \vec{v}_k\}$ note $2\cdot\vec{0} + 0\vec{v}_1 + .. + 0\vec{v}_k = \vec{0}$ but not all the scalars are zero.

f) true We need to check for lin. ind. of the 3 vectors.

$$\left[\begin{array}{ccc|c} i & 0 & 1 & 0 \\ 0 & i & i & 0 \end{array}\right] \xrightarrow[-iR_2]{-iR_1} \left[\begin{array}{ccc|c} 1 & 0 & -i & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

so $i\begin{bmatrix} i \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ i \end{bmatrix} + \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ but the scalars are not all zero

11) Given $V = \{\vec{0}\}$ we'd like to show $S = \{\vec{0}\}$ is not a basis.

Notice $1\cdot\vec{0} = \vec{0}$ so we've written a linear combo of vectors from S with scalars that are not all zero but whose sum is $\vec{0}$. Thus S is lin. dep.

Since a basis is lin. independent $\vec{0} \notin$ a basis

Since V consists of only the $\vec{0}$ element the basis must be empty $\Rightarrow \dim V = 0$.

12) Let $\vec{w}_1, \dots \vec{w}_r$ be lin. indep vectors in W.
Let $\vec{w} \in W$ and $\vec{w} \notin \text{span}\{\vec{w}_1, \dots \vec{w}_r\}$
We want to show the set $\{\vec{w}_1, \dots \vec{w}_r, \vec{w}\}$ is lin. indep.

I'll use a proof by contradiction: That is I'll assume the hypotheses and the opposite of the conclusion & bump into a contradiction.

Assume towards contradiction that $\{\vec{w}_1, \dots \vec{w}_r, \vec{w}\}$ is is not linearly indepent.
This means there exist scalars $c_1, \dots c_r, c$ that are not all zero with the property $c_1\vec{w}_1 + \dots + c_r\vec{w}_r + c\vec{w} = \vec{0}$
Subtracting $c\vec{w}$ from both sides we see
$$c_1\vec{w}_1 + \dots + c_r\vec{w}_r = -c\vec{w}$$

If $c \neq 0$ we could mult both sides by $-1/c$
$\Rightarrow -c_1/c\,\vec{w}_1 + \dots + -c_r/c\,\vec{w}_r = \vec{w}$
$\Rightarrow \vec{w} \in \text{span}\{\vec{w}_1, \dots \vec{w}_r\}$ but that contradicts our assumption that $\vec{w} \notin \text{span}\{\vec{w}_1, \dots \vec{w}_r\}$.
If $c = 0$ then we'd have
$$c_1\vec{w}_1 + \dots + c_r\vec{w}_r = \vec{0}$$
where not all the c_i are zero but that contradicts our assumption that $\{\vec{w}_1, \dots \vec{w}_r\}$ is lin. indep.
Thus such c_i cannot exist $\Rightarrow \{\vec{w}_1, \dots \vec{w}_r, \vec{w}\}$ is lin. ind.

20) Let $T: V \to W$ be a lin. operator s.t. range $T = W$ & Ker $T = \{\vec{0}\}$. Let $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ be a basis of V.

To show $\{T(\vec{v}_1), \ldots T(\vec{v}_n)\}$ is a basis of W we show it is both a spanning set + lin. indep.

Spanning set: Let $\vec{w} \in W$, we want to find scalars $d_1, \ldots d_n$ so that $\vec{w} = d_1 T(\vec{v}_1) + \ldots + d_n T(\vec{v}_n)$.

Since range $T = W$, there exists a $\vec{v} \in V$ so that $T(\vec{v}) = \vec{w}$.
Since $\{\vec{v}_1, \ldots \vec{v}_n\}$ forms a basis for V there exists scalars $c_1, \ldots c_n$ so we can write $\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$.

Consider
$\vec{w} = T(\vec{v}) = T(c_1\vec{v}_1 + \ldots + c_n\vec{v}_n)$ b/c T is linear
$= c_1 T(\vec{v}_1) + \ldots + c_n T(\vec{v}_n)$
Thus we have what we needed.

Linear independent: Recall the def. of lin. indep.
We want to show if we are given $a_1 T(\vec{v}_1) + \ldots + a_n T(\vec{v}_n) = \vec{0}$ that $0 = a_1 = \ldots = a_n$.

Notice since T is linear
$\vec{0} = a_1 T(\vec{v}_1) + \ldots + a_n T(\vec{v}_n) = T(a_1\vec{v}_1 + \ldots + a_n\vec{v}_n)$
So $a\vec{v}_1 + \ldots + a_n\vec{v}_n \in \text{Ker}\, T = \{0\}$
Since $\vec{v}_1, \ldots \vec{v}_n$ is a basis
$a\vec{v}_1 + \ldots + a_n\vec{v}_n = 0 \Rightarrow a_1 = 0 = a_2 = \ldots = a_n$
Thus we have our conclusion

23) Let $T: V \to W$ be a linear operator where V is fin. dim space & U is a subspace of V. Show $\dim T(U) \leq \dim(U)$.

Let $\{\vec{u}_1, \ldots \vec{u}_n\}$ be a basis of U. So $\dim(U) = n$

I claim $T(U) = \text{image } T = \text{span}\{T(\vec{u}_1), \ldots T(u_n)\}$. Once this is shown S. algorithm will be able to find a basis by eliminating vectors thus $\dim T(U) \leq n = \dim(U)$.

Pf of claim: Let $\vec{w} \in T(U)$. We need to show that there exist scalars $a_1 \ldots a_n$ that will allow us to write $\vec{w} = a_1 T(\vec{u}_1) + \ldots + a_n T(\vec{u}_n)$

Since $\vec{w} \in T(U)$ by definition of image there exists a $\vec{u} \in U$ so that $T(\vec{u}) = \vec{w}$.
$\{\vec{u}_1, \ldots \vec{u}_n\}$ is a basis of U so there are scalars $b_1 \ldots b_n$ such that $\vec{u} = b_1\vec{u}_1 + \ldots + b_n\vec{u}_n$

Consider then

$$\vec{w} = T(\vec{u}) = T(b_1\vec{u}_1 + \ldots + b_n\vec{u}_n) \quad \text{by linearity}$$
$$= b_1 T(\vec{u}_1) + \ldots + b_n T(\vec{u}_n)$$

thus $\vec{w} \in \text{span}\{T(\vec{u}_1), \ldots T(\vec{u}_n)\}$
$\Rightarrow \{T(\vec{u}_1), \ldots T(\vec{u}_n)\}$ is a spanning set //