§3.4 #~~1ab~~, ~~3ab~~, ~~5ab~~, ~~7a~~, ~~9~~, ~~10~~, ~~11~~, ~~12~~, 14 10.12.16

1a) $\left[\begin{array}{cccc|c} 2 & -1 & 0 & 3 & 0 \\ 4 & -2 & 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & -1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right]$ $\{(x_1, x_2, x_3, x_4) \mid 2x_1 - x_2 + 3x_4 = 0, x_3 - 3x_4 = 0\}$ $x_2, x_4 \in \mathbb{R}$

$\{(\frac{1}{2}x_2 - \frac{3}{2}x_4, x_2, 3x_4, x_4) \mid x_2, x_4 \in \mathbb{R}\}$ so $\left\{\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ 3 \\ 1 \end{bmatrix}\right\}$

b) $\left[\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\{(x_1, x_2) \mid x_1 + 4x_2 = 0\} = \{(-4x_2, x_2) \mid x_2 \in \mathbb{R}\}$

$\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}$

3a) span $\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}\right\}$ = span $\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ by above rref

so $\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

b) span $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix}\right\}$ = span $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ basis: $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

5) a) span $\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 3 \end{bmatrix}\right\}$ note $\begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 0 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \\ 0 & -3 \end{bmatrix}$ lin. ind basis: $\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 3 \end{bmatrix}\right\}$

b) span $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \end{bmatrix}\right\}$ = span $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$ basis: $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$

7a) nullspace of I-A $I - A = \begin{bmatrix} .5 & 0 & -1 \\ -.5 & .5 & 0 \\ 0 & -.5 & 1 \end{bmatrix}$

$I - A \xrightarrow{R_2 + R_1 \to R_2} \begin{bmatrix} .5 & 0 & -1 \\ 0 & .5 & -1 \\ 0 & -.5 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} .5 & 0 & -1 \\ 0 & .5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{10}{5}R_2]{\frac{10}{5}R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

sol set $= \{(x_1, x_2, x_3) \mid x_1 = 2x_3, x_2 = 2x_3, x_3 \in \mathbb{R}\}$

$= \{(2x_3, 2x_3, x_3) \mid x_3 \in \mathbb{R}\}$

$=$ span $\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}$

we are looking for $\vec{y} \in$ span $\left\{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}$ whose entries are nonneg & sum to 1

ie looking for scalar c so that $\begin{bmatrix} c2 \\ c2 \\ c1 \end{bmatrix}$ has the property

that $2c + 2c + c = 1 \Rightarrow 5c = 1 \Rightarrow c = 1/5$

try $\vec{y} = 1/5 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 1/5 \end{bmatrix}$!!

9) a) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \longmapsto \begin{bmatrix}x_1-2x_2+x_3\\x_1+x_2+x_3\\2x_1-x_2+2x_3\end{bmatrix} = \begin{bmatrix}1&-2&1\\1&1&1\\2&-1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$ ker A

note that there is a matrix we can assoc. with T & thus we can use the identifications listed on pg 199 that will make the computations easier

$$A := \begin{bmatrix}1&-2&1\\1&1&1\\2&-1&2\end{bmatrix} \to \begin{bmatrix}1&-2&1\\0&3&0\\0&3&0\end{bmatrix} \to \begin{bmatrix}1&-2&1\\0&1&0\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}$$

$$\ker T = N(A) = \{(x_1,x_2,x_3) \mid x_1=-x_3,\ x_2=0,\ x_3\in\mathbb{R}\}$$
$$= \{(-x_3,0,x_3) \mid x_3\in\mathbb{R}\}$$
$$= \operatorname{span}\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$ not one-to-one

$$\operatorname{range} T = C(A) = \operatorname{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$$ by rref above

$$= \operatorname{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\end{bmatrix}\right\}$$ not **onto**

dim range $T = 2$ but dim target $= 3$

also note $[0,0,1]^T$ is not in the range

b/c $$\left[\begin{array}{cc|c}1&-2&0\\1&1&0\\2&-1&1\end{array}\right] \to \left[\begin{array}{cc|c}1&-2&0\\0&3&0\\0&3&1\end{array}\right] \to \left[\begin{array}{cc|c}1&-2&0\\0&3&0\\0&0&1\end{array}\right]$$ incons't

b) $T : P_2 \to \mathbb{R}$

$ax^2+bx+c \longmapsto a(1)^2+b(1)+c = a+b+c$

I could use the identification with P_2 and $\mathbb{R}^3$ but I'll try doing this with just the definition.

just fine →

$$\ker T = \{ax^2+bx+c \in P_2 \mid a(1)^2+b(1)+c=0\}$$
$$= \{ax^2+bx+c \in P_2 \mid a=-b-c\} = \{(-b-c)x^2+bx+c \mid c,b\in\mathbb{R}\}$$
$$= \{(-x^2+x)b+(-x^2+1)c \mid c,b\in\mathbb{R}\}$$
$$= \operatorname{span}\{-x^2+x,\ -x^2+1\}$$ so not one-to-one

range $T = \{ z \in \mathbb{R} \mid \exists\ ax^2+bx+c \in P_2$ with $T(ax^2+bx+c) = z\}$
$= \{ a+b+c \mid a,b,c \in \mathbb{R}\}$
$= \text{span}\{1\}$

is onto

10) a) $T: P_2 \to P_3$ and
$a+bx+cx^2 \mapsto ax + \frac{b}{2}x^2 + \frac{c}{3}x^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto \begin{bmatrix} 0 \\ a \\ b/2 \\ c/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Given 9b I think I'll record the polynomials as vectors and find a matrix that records my linear transform as matrix mult from left.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\text{Ker} T} = N(A) = \{(y_1,y_2,y_3) \mid y_1 = 0 = y_2 = y_3\} = 0$ T is one-to-one

$\underline{\text{range} T} = C(A) = \text{span}\{x, x^2, x^3\}$ T is not onto.

again an argument of dim or the fact $2 \in P_3$ but $2 \notin$ range T.

b) $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 2x_2 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{Let } A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Started (+1)
Ker & 1-1 (+1)
range & onto (+1)

$\underline{\text{Ker} T} = N(A) = \{(x_1,x_2,x_3) \mid x_2 = 0 = x_3,\ x_1 \in \mathbb{R}\} = \{(x_1,0,0) \mid x_1 \in \mathbb{R}\}$
$= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

not one-to-one

$\underline{\text{range} T} = C(A) = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right\}.$

is onto

11) $T: V \to \mathbb{R}^2$ $\quad T(\vec{v}_1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad T(\vec{v}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad T(\vec{v}_3) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

where $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ every vector in V can be expressed uniquely with a set of coordinates c_1, c_2 & c_3.
Thus

- $\operatorname{Ker} T = \{\vec{v} \in V \mid T(\vec{v}) = 0\}$

$$= \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \in V \mid T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = 0\}$$

by lin. of T $= \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \in V \mid c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = 0\}$

$$= \{\text{"} \qquad \text{"} \mid c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0\}.$$

we are thus looking for all solutions to the lin sys.

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

So sol set is $\{(c_1, c_2, c_3) \mid c_1 = c_3,\ c_2 = -c_3,\ c_3 \in \mathbb{R}\}$

$\{(c_3, -c_3, c_3) \mid c_3 \in \mathbb{R}\} := S$

Any set of coordinates from S will thus

$\Rightarrow c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0$

ie the vector assoc. with these coordinates is in the kernel

$\operatorname{Ker} T = \{c_3\vec{v}_1 - c_3\vec{v}_2 + c_3\vec{v}_3 \mid c_3 \in \mathbb{R}\}$ $\qquad$ not one-to-one

$= \operatorname{span}\{\vec{v}_1 - \vec{v}_2 + \vec{v}_3\}$

- range $T = \operatorname{span}\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} = \operatorname{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}\}$

$= \operatorname{span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$

note $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ forms a basis for $\mathbb{R}^2$ $\qquad$ is onto

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

started (+1)
kernel 1-1 (+1)
range & onto (+1)

12) $T: V \to \mathbb{R}^3$ $\quad T(\vec{v}_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad T(\vec{v}_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad T(\vec{v}_3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

where $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V

We'll use the same trick we did in #11.

$$\operatorname{Ker} T = \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \mid T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = \vec{0}\}$$
$$= \{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \mid c_1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \vec{0}\}$$

looking at the lin. system

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $c_1 = -c_3$, $c_2 = c_3$ if $c_1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$

Thus

$\operatorname{Ker} T = \operatorname{span}\{-\vec{v}_1 + \vec{v}_2 + \vec{v}_3\} = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\} \subset V$ is not one-to-one

$$\operatorname{range} T = \operatorname{span}\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$$
$$= \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$
$$= \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} \subset \mathbb{R}^3$$

The above rref shows the third vector is redundant. Thus a dim. argument $\Rightarrow$ not onto

16) Show that if $T: V \to W$ is a linear operator, then the kernel of T is a subspace of V.

started (+1)
looked for 3 (+1)
got it (+1)
(+1)

We need to check that the 3 conditions to be a subspace hold.

(1) Note $\vec{0}_V \in V$ is in the kernel of T since

$T(\vec{0}_V) = T(0 \cdot \vec{0}_V) = 0\, T(\vec{0}_V)$ since T is linear
$= \vec{0}_W$ b/c $0 \cdot \vec{w} = \vec{0}$ $\forall \vec{w} \in W$

(2) Let $\vec{v}_1, \vec{v}_2 \in \operatorname{Ker} T$, we need to show $\vec{v}_1 + \vec{v}_2 \in \operatorname{Ker} T$.

$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ since T is linear
$= \vec{0}_W + \vec{0}_W$ since $\vec{v}_1, \vec{v}_2 \in \operatorname{Ker} T$
$= \vec{0}_W$ so $\vec{v}_1 + \vec{v}_2 \in \operatorname{Ker} T$.

(3). Let c be a scalar and $\vec{v}_1 \in \operatorname{Ker} T$.
We need to show $c\vec{v}_1 \in \operatorname{Ker} T$.

$$\begin{aligned} T(c\vec{v}_1) &= cT(\vec{v}_1) && \text{since } T \text{ is linear} \\ &= c\vec{0}_W && \text{since } \vec{v}_1 \in \operatorname{Ker} T \\ &= \vec{0}_W \end{aligned}$$

Thus $c\vec{v}_1 \in \operatorname{Ker} T$.

So $\operatorname{Ker} T$ is a subspace of V. //