$\S 3.3$ # ~~3~~, 4, ~~5~~, ~~7~~, ~~9~~, ~~11~~, ~~13~~, ~~20~~, ~~25~~

in back: 3, 5, 7, 9, 11, 13, 20
grade: 4 25

3a) $\{1, x, x^2, x^3\}$ is linearly independent.

Justification: if c_1, c_2, c_3, c_4 where such that

$c_1 \cdot 1 + c_2 x + c_3 x^2 + c_4 x^3 = 0 \quad \Rightarrow c_1 = 0 = c_2 = c_3 = c_4$

b) $\{1+x, 1+x^2, 1+x^3\}$

If c_1, c_2, c_3 where such that

$c_1(1+x) + c_2(1+x^2) + c_3(1+x^3) = 0$

$\Rightarrow (c_1+c_2+c_3) + c_1 x + c_2 x^2 + c_3 x^3 = 0 + 0x + 0x^2 + 0x^3$

Since the coef. on each term must match

$c_3 = 0 = c_2 = c_1$

$\therefore$ linearly independent

c) $\{1-x^2, 1+x, 1-x-2x^2\}$

If c_1, c_2, c_3 where such that

$c_1(1-x^2) + c_2(1+x) + c_3(1-x-2x^2) = 0$

$\Rightarrow (c_1+c_2+c_3) + (c_2-c_3)x + (-c_1-2c_3)x^2 = 0 + 0x + 0x^2$

Since the coef. on each term must match

$-c_1 \quad -2c_3 = 0$
$c_2 - c_3 = 0$
$c_1 + c_2 + c_3 = 0$

linear sys. of equations $\Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $c_2 = c_3$ and $c_1 + 2c_3 = 0$

$\{(-2c_3, c_3, c_3) \mid c_3 \in \mathbb{R}\}$.

Let $c_3 = 1$ & notice

$-2(1-x^2) + (1+x) + (1-x-2x^2) = -2 + \cancel{2x^2} + 1 + \cancel{x} + 1 - \cancel{x} - \cancel{2x^2} = 0$

So $-(1+x) + (1-x-2x^2) = 2(1-x^2)$

linearly dependent

d) $\{x^2-x^3, x, -x+x^2+3x^3\}$ same approach as above

$c_1 x^2 - c_1 x^3 + c_2 x - c_3 x + c_3 x^2 + 3c_3 x^3 = 0$

$(c_2-c_3)x + (c_1+c_3)x^2 + (3c_3-c_1)x^3 = 0 \quad \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\Rightarrow c_1 = 0 = c_2 = 0 = c_3$ linearly indep.

4) a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ looking for $c_1, c_2 \ni c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

linear sys. of equations.

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so $c_1 = 0 = c_2$ is the only solution.

linearly indep

b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\}$ I'm going to use that trick from thm 3.7

$$\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & -1 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} -1 & -1 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} -1 & -1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \underline{0}$$ linearly dependent

unfortunately - this doesn't actually tell me the redundancies so I still have to do the work I did for a.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so $\{(-c_4, 2c_4, 0, c_4) \mid c_4 \in \mathbb{R}\}$ should be a sol. set

let $c_4 = 1$ then

$$-\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2+0-1 \\ -1+0+0+1 \\ 0+2+0-2 \\ 0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ ✓ <u>lin. dep</u>

c) $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 6 \end{bmatrix} \right\}$ $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ so $\{(-c_3, -c_3, c_3) \mid c_3 \in \mathbb{R}\}$

5) a) find c_1, c_2 so that $c_1\begin{bmatrix}2\\1\end{bmatrix}+c_2\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$ ie $\begin{bmatrix}2&2\\1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$

$$\left[\begin{array}{cc|c}2&2&-1\\1&-1&1\end{array}\right]\to\left[\begin{array}{cc|c}1&-1&1\\2&2&-1\end{array}\right]\xrightarrow{R_2-2R_1}\left[\begin{array}{cc|c}1&-1&1\\0&4&-3\end{array}\right]\to\left[\begin{array}{cc|c}1&-1&1\\0&1&-3/4\end{array}\right]\to\left[\begin{array}{cc|c}1&0&1/4\\0&1&-3/4\end{array}\right]$$

CK: $\frac14\begin{bmatrix}2\\1\end{bmatrix}+-\frac34\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}\frac12-\frac32\\\frac14+\frac34\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$ ✓

$$\begin{bmatrix}-1\\1\end{bmatrix}_{\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}2\\-1\end{bmatrix}\}}=\left(\tfrac14,-\tfrac34\right)$$

b) find c_1,c_2,c_3 s.t. $c_1(1+x)+c_2(x+x^2)+c_3(1-x)=2+x^2$

$$(c_1+c_3)+(c_1+c_2-c_3)x+c_2x^2=2+x^2$$

so
$$\begin{aligned}c_1+\quad c_3&=2\\c_1+c_2-c_3&=0\\c_2\quad&=1\end{aligned}\quad\text{-or-}\quad\begin{bmatrix}1&0&1\\1&1&-1\\0&1&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&0&1&2\\1&1&-1&0\\0&1&0&1\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&1&2\\0&1&-2&-2\\0&1&0&1\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&1&2\\0&1&0&1\\0&0&-2&-3\end{array}\right]\to\left[\begin{array}{ccc|c}1&0&1&2\\0&1&0&1\\0&0&1&3/2\end{array}\right]$$

$$\to\left[\begin{array}{ccc|c}1&0&0&1/2\\0&1&0&1\\0&0&1&3/2\end{array}\right]$$

CK: $\frac12(1+x)+(x+x^2)+\frac32(1-x)=\frac12+\cancel{\frac12x}+\cancel{x}+x^2+\frac32-\cancel{\frac32x}=2+x^2$ ✓

so $[2+x^2]_{\{1+x,\,x+x^2,\,1-x\}}=\left(\frac12,1,\frac32\right)$

c) find c_1,c_2,c_3 s.t. $c_1\begin{bmatrix}0&1\\1&0\end{bmatrix}+c_2\begin{bmatrix}1&0\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}a&b\\b&c\end{bmatrix}$

so we need
$$\begin{aligned}0\cdot c_1+1\cdot c_2+0\cdot c_3&=a &&\Rightarrow c_2=a\\1\cdot c_1+0\cdot c_2+0\cdot c_3&=b &&\Rightarrow c_1=b\\1\cdot c_1+0\cdot c_2+0\cdot c_3&=b &&\Rightarrow c_1=b \quad \text{good}\\0\cdot c_1+0\cdot c_2+1\cdot c_3&=c &&\Rightarrow c_3=c\end{aligned}$$

CK: ✓

so $\begin{bmatrix}a&b\\b&c\end{bmatrix}_{\{\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\}}=(b,a,c)$

d) find $c_1,c_2\in\mathbb{C}$ s.t. $c_1\begin{bmatrix}2+i\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\i\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ -or- $\begin{bmatrix}2+i&-1\\1&i\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

$-1-(2+i)i=\cancel{1}-2i+\cancel{1}$

$1-(2+i)2=1-4-2i$

$$\left[\begin{array}{cc|c}2+i&-1&1\\1&i&2\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{cc|c}1&i&2\\2+i&-1&1\end{array}\right]\xrightarrow{R_2-(2+i)R_1}\left[\begin{array}{cc|c}1&i&2\\0&-2i&-3-2i\end{array}\right]\xrightarrow{\frac{-1}{2i}R_2}\left[\begin{array}{cc|c}1&i&2\\0&1&1-\frac32i\end{array}\right]$$

$\frac{-3-2i}{-2i}\frac{(2i)}{(2i)}=\frac{-6i+4}{4}$

$$\xrightarrow{R_1-iR_2}\left[\begin{array}{cc|c}1&0&\frac12-i\\0&1&1-\frac32i\end{array}\right]$$

Check: $\left(\frac12-i\right)\begin{bmatrix}2+i\\1\end{bmatrix}+\left(1-\frac32i\right)\begin{bmatrix}-1\\i\end{bmatrix}=\begin{matrix}1+\frac12i-2i+\cancel{1}-\cancel{1}+\frac32i\ ✓\\\frac12-\cancel{i}+\cancel{i}+\frac32\end{matrix}$

$2-i(1-\frac32i)=2-i-\frac32$

So $\begin{bmatrix}1\\2\end{bmatrix}_{\{\begin{bmatrix}2+i\\1\end{bmatrix},\begin{bmatrix}-1\\i\end{bmatrix}\}}=\left(\frac12-i,\,1-\frac32i\right)$

7) a) determine if $\exists\ c_1 \,\&\, c_2$ s.t. $c_1\begin{bmatrix}1\\0\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}2\\1\\2\end{bmatrix}$

$$\left[\begin{array}{cc|c}1&1&2\\0&-1&1\\1&1&2\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1&1&2\\0&+1&-1\\0&0&0\end{array}\right]\rightarrow\left[\begin{array}{cc|c}1&0&3\\0&1&-1\\0&0&0\end{array}\right] \quad \text{yup.}$$

CK: $3\begin{bmatrix}1\\0\\1\end{bmatrix}-\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}2\\1\\2\end{bmatrix}$ ✓

b) note that $e_1 \notin \text{span}\{\vec{u}_1,\vec{u}_2\}$

to find a basis we need to make sure our set spans $\mathbb{R}^3$.

note $\mathbb{R}^3 = \text{span}\{e_1,e_2,e_3\}$.

We'd like to make sure $\vec{u}_1$ & $\vec{u}_2$ are included in the set

so try running the algorithm introduced in class on

$\{\vec{u}_1,\vec{u}_2,e_1,e_2,e_3\}$ → note this set spans $\mathbb{R}^3$

we just need to make sure

it is minimal

the algorithm will return the set $\{\vec{u}_1,\vec{u}_2,e_1\}$.

note: there are many vectors you could have

chosen from $\mathbb{R}^3$ that would have worked

9) I assume that the given relation $\vec{v}_2+2\vec{v}_3=\vec{0}$ is

the only relation between $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

Thus $\vec{v}_1$ cannot be written as a linear combo of $\vec{v}_2$ & $\vec{v}_3$

$\therefore$

$\{\vec{v}_1,\vec{v}_2\}$ and $\{\vec{v}_1,\vec{v}_3\}$

11) I'll make use of Thm. 3.7 again & find what conditions

on $c \Rightarrow \begin{bmatrix}1&2&3c+1\\1&c&3\\c&4&-4\end{bmatrix}$ is invertible.

So we need c so that $0\neq\begin{vmatrix}1&2&3c+1\\1&c&3\\c&4&-4\end{vmatrix}=\begin{vmatrix}1&2&3c+1\\0&c-2&2-3c\\c&4&-4\end{vmatrix}=1\begin{vmatrix}c-2&2-3c\\4&-4\end{vmatrix}+c\begin{vmatrix}2&3c+1\\c-2&2-3c\end{vmatrix}$

So $0\neq(c-2)(-4)-4(2-3c)+c\big(2(2-3c)-(c-2)(3c+2)\big)=-4c+\cancel{8}\cancel{-8}+12c+c(4-6c-(c-2)(3c+2))$

So $0\neq 8c+c(4-6c-(3c^2+2c-6c-4))=8c+c(4-6c-3c^2+4c+4)$

$=8c+8c-2c^2-3c^3$

$\frac{2\pm\sqrt{4+4\cdot3(16)}}{2\cdot-3}$

$\frac{2\pm 14}{-6}$

So $0\neq c(16-2c-3c^2) \Rightarrow c\neq 0, -16/6, 2$

$\Rightarrow c\neq 0, -8/3, 2$

13) note $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A - B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = -A + B + C = -\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A - C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

but

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = e_{11} \notin \text{span}\{A, B, C\}$ so e_{11}

22) Prove that a list of vectors $\{\vec{v}_1, \ldots \vec{v}_n\}$ with repeated vectors is linearly dependent.

Let $\vec{v}_a, \vec{v}_b \in \{\vec{v}_1, \ldots \vec{v}_n\}$ such that $a \neq b$ but $\vec{v}_a = \vec{v}_b$
(ie name the repeated vectors $\vec{v}_a$ + $\vec{v}_b$).
Consider the linear combination

$$0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_{a-1} + 2\vec{v}_a + 0\vec{v}_{a+1} + \ldots + 0\vec{v}_{b-1} + {}^-2\vec{v}_b + 0\vec{v}_{b+1} + \ldots 0\vec{v}_n$$

This equals the $\vec{0}$ but there are 2 nonzero coef. Thus $\{\vec{v}_1, \ldots \vec{v}_n\}$ is linearly dependent

25) • Let $\{\vec{v}_1, \ldots \vec{v}_n\}$ be a linearly dependent set. Then

There exists scalars $c_1, \ldots c_n$, not all zero with the property
$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$.
Apply T to both sides:

$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) = T(\vec{0})$ since T is linear
$c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots c_nT(\vec{v}_n) = T(0 \cdot \vec{0})$
recall that not all $c_1 \ldots c_n$ were zero so we have
$c_1T(\vec{v}_1) + \ldots + c_nT(\vec{v}_n) = \vec{0}$
$\Rightarrow \{T(\vec{v}_1), T(\vec{v}_2), \ldots T(\vec{v}_n)\}$ is linearly dependent

• Let $T: \mathbb{R}^2 \to \mathbb{R}$
$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto 0$

note the set $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \subset \mathbb{R}^2$ is linearly independent but
$\left\{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \{\vec{0}, \vec{0}\}$ which is linearly dependent.