§3.2 #~~3-5, 8, 11-15, 17-20~~

let $(a,b,c),(\alpha,\beta,\gamma)\in W$ and $d\in\mathbb{R}$

3) $W=\{(a,b,c)\mid 2a-b+c=0\}\leq\mathbb{R}^3$

(1) $(0,0,0)$ is $\ni$ $2\cdot0-0+0=0$ ✓

(2) $(a+\alpha,b+\beta,c+\gamma)$ $\quad 2(a+\alpha)-(b+\beta)+c+\gamma=(2a-b+c)+2\alpha-\beta+\gamma=0$ ✓

(3) (da,db,dc) $\quad 2da-db+dc=d(2a-b+c)=d\cdot0=0$ ✓

subspace

let $\begin{bmatrix}a&b&0\\b&a&0\end{bmatrix},\begin{bmatrix}\alpha&\beta&0\\\beta&\alpha&0\end{bmatrix}\in W$ and $d\in\mathbb{R}$

4) $W=\left\{\begin{bmatrix}a&b&0\\b&a&0\end{bmatrix}\middle| a,b\in\mathbb{R}\right\}\subset\mathbb{R}^{2,3}=Mat_{2\times3}(\mathbb{R})$

(1) $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\in W$

(2) $\begin{bmatrix}a+\alpha&b+\beta&0\\b+\beta&a+\alpha&0\end{bmatrix}\in W$ by closure of $\mathbb{R}$

(3) $\begin{bmatrix}da&db&0\\db&da&0\end{bmatrix}\in W$ by closure of $\mathbb{R}$ ✓

subspace

5) $W=\{f:[0,1]\to\mathbb{R}\mid f \text{ is con't } \& \ f(1)+f(\tfrac12)=0\}\subset C[0,1]$

let $g,h\in W$ and $d\in\mathbb{R}$

(1) Recall $\vec{0}\in C$ is defined by $\vec{0}(x)=0$

$\vec{0}(1)+\vec{0}(\tfrac12)=0+0=0$ ✓ so $\vec{0}\in W$

(2) Math 251 $\Rightarrow$ $g+h$ is con't.

$(g+h)(1)+(g+h)(\tfrac12)=g(1)+h(1)+g(\tfrac12)+h(\tfrac12)$ by comm of $\mathbb{R}$

$=\underbrace{g(1)+g(\tfrac12)}_{0}+\underbrace{h(1)+h(\tfrac12)}_{0}$ b/c $g,h\in W$

$=0+0=0$

$\Rightarrow g+h\in W$

(3) Math 251 $\Rightarrow$ dg is con't

$dg(1)+dg(\tfrac12)=d(g(1))+d(g(\tfrac12))=d(g(1)+g(\tfrac12))$

$=d(0)=0$

$\Rightarrow dg\in W$

subspace

let $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \in W$ and $d \in \mathbb{R}$

8) $W = \left\{ \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \mid a,b,c \in \mathbb{R} \right\} \subset \mathbb{R}^{2,2} = \mathrm{Mat}_{2\times 2}(\mathbb{R})$

(1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ ✓

(2) $\begin{bmatrix} a+\alpha & b+\beta \\ -b-\beta & c+\gamma \end{bmatrix} = \begin{bmatrix} a+\alpha & b+\beta \\ -(b+\beta) & c+\gamma \end{bmatrix} \in W$ ✓

(3) $\begin{bmatrix} d\alpha & db \\ d(-b) & dc \end{bmatrix} = \begin{bmatrix} d\alpha & db \\ -db & dc \end{bmatrix} \in W$ ✓

subspace

These are not skew-symmetric?

Skew symmetric $= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \mid b \in \mathbb{R} \right\}$

then $A^T = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = -A$.

11) Show $\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Let $\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} =: \mathrm{span}\{\vec{e}_1, \vec{e}_2\}$ & $\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} = \mathrm{span}\{\vec{v}_1, \vec{v}_2\}$

note $\vec{v}_1 = 1 \cdot \vec{e}_1 + 0 \cdot \vec{e}_2$

& $\vec{v}_2 = -2\vec{e}_1 + \vec{e}_2$

so $\mathrm{span}\{\vec{v}_1, \vec{v}_2\} \subseteq \mathrm{span}\{\vec{e}_1, \vec{e}_2\}$.

note $\vec{e}_1 = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$

$\vec{e}_2 = 2\vec{v}_1 + 1 \cdot \vec{v}_2$

so $\mathrm{span}\{\vec{e}_1, \vec{e}_2\} \subseteq \mathrm{span}\{\vec{v}_1, \vec{v}_2\}$

Double containment $\Rightarrow$ the two sets are equal.

12) Show $\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}0\\-1\\-1\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$

We'll again show double inclusion. Denote the vectors on the left with $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$ while denoting vectors on the right with $\vec{u}_1$ & $\vec{u}_2$.

Note $\vec{u}_1 = \begin{bmatrix}0\\-1\\-1\end{bmatrix} = 0\vec{v}_1 - 1\cdot\vec{v}_2 + 0\vec{v}_3$ and

$\vec{u}_2 = \begin{bmatrix}1\\2\\2\end{bmatrix} = 0\vec{v}_1 + 1\cdot\vec{v}_2 + 1\cdot\vec{v}_3$ so $\text{span}\{\vec{u}_1,\vec{u}_2\} \subseteq \text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$

Note $\vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} = 2\vec{u}_1 + 1\cdot\vec{u}_2$

$\vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} = 0\vec{u}_1 - 1\cdot\vec{u}_2$

motivation for following

$\vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} = 1\vec{u}_1 + 1\vec{u}_2$

I don't think I can do the third in my head so looking for c_1 & c_2 $\ni$

$$\vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} = c_1\begin{bmatrix}0\\-1\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}0&1\\-1&2\\-1&2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

linear system:

$$\left[\begin{array}{cc|c}0&1&1\\-1&2&1\\-1&2&1\end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{cc|c}-1&2&1\\0&1&1\\-1&2&1\end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{cc|c}1&-2&-1\\0&1&1\\-1&2&1\end{array}\right]$$

$$\xrightarrow{R_3+R_1\to R_3} \left[\begin{array}{cc|c}1&-2&-1\\0&1&1\\0&0&0\end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&0&0\end{array}\right]$$

so $c_1 = 1 = c_2$ //

so $\text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\} \subseteq \text{span}\{\vec{u}_1,\vec{u}_2\}$

$\therefore$ we have double inclusion

$\therefore \text{span}\{\vec{u}_1,\vec{u}_2\} = \text{span}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$.

13 Let ax^2+bx+c be an arbitrary element of P_2. If we show ax^2+bx+c is in the span of vectors given $\Rightarrow P_2$ is contained in the given span. Since the given spans are contained in P_2 we'll have double containment.

a) Find $c_1, c_2, c_3 \ni c_1(1) + c_2(1+x) + c_3(x^2) = ax^2 + bx + c$

$\Rightarrow c_1 + c_2 + c_2 x + c_3 x^2 = ax^2 + bx + c$

$\Rightarrow c_3 = a \quad c_2 = b$ and $c_1 = c - c_2 = c - b$

note

$(c-b)(1) + (b)(1+x) + ax^2 = c - \cancel{b} + \cancel{b} + bx + ax^2 = c + bx + ax^2$ ✓

so $ax^2 + bx + c \in \text{span}\{1, 1+x, x^2\} \Rightarrow P_2 = \text{span}\{1, 1+x, x^2\}$.

b) Find $c_1, c_2, c_3 \ni c_1(x) + c_2(4x - 2x^2) + c_3 x^2 = ax^2 + bx + c$

note: the polynomial $1 \in P_2$ but $\nexists$ coef $\ni c_1(x) + c_2(4x - 2x^2) + c_3 x^2 = 1$

to justify this a bit more - record a given polynomial in $\mathbb{R}^3$ by assigning the i^{th} row to contain the coef on the $i-1^{th}$ deg. term. for ex x corresponds to $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $4x - 2x^2$ corresponds to $\begin{bmatrix}0\\4\\-2\end{bmatrix}$. (just as we did in class)

We are thus looking for $(c_1, c_2, c_3)^T \ni$

$\begin{bmatrix}0&0&0\\1&4&0\\0&-2&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ notice that it is already inconsistent $\therefore \nexists \ (c_1, c_2, c_3)^T$

$\Rightarrow P_2 \neq \text{span}\{x, 4x - 2x^2, x^2\}$

c) Another way we could do this problem is to recognize $P_2 = \text{span}\{1, x, x^2\}$ so we could do the same things we did for 1) & 2) & see if these standard vectors can be written as a linear combination of the given vectors.

note

$1 = 0 \cdot (1 + x + x^2) + 0 \cdot (1+x) + \frac{1}{3}(3)$

$x = 0(1 + x + x^2) + 1(1+x) - \frac{1}{3}(3)$

$x^2 = 1 \cdot (1 + x + x^2) - 1(1+x) + 0 \cdot 3$

$\Rightarrow P_2 = \text{span}\{1, x, x^2\} \subseteq \text{span}\{1 + x + x^2, 1+x, 3\}$

$\Rightarrow P_2 = \text{span}\{1 + x + x^2, 1+x, 3\}$.

13d) note we can't write $x \in P_2$ as a lin. combo of $1-x^2$ and 1 so $P_2 \neq \text{span}\{1-x^2, 1\}$.

17) a) $U \cap V = \{w \in W \mid w \in U \text{ and } w \in V\}$.
We prove the three properties.
(1) Let $\vec{0} \in W$ we need to show $\vec{0} \in U \cap V$.
Since U is a subspace $\vec{0} \in U$.
Since V is a subspace $\vec{0} \in V$ $\Big\} \Rightarrow \vec{0} \in U \cap V$.
(2) Let $\vec{u}, \vec{v} \in U \cap V$.
Since $\vec{u}, \vec{v} \in U \cap V \Rightarrow \vec{u}, \vec{v} \in U$. Since U is a subspace $\vec{u} + \vec{v} \in U$. Similarly $\vec{u} + \vec{v} \in V$ $\therefore \vec{u} + \vec{v} \in U \cap V$.
(3) Let $\vec{u} \in U \cap V$ and $\alpha \in \mathbb{F}$.
Since $\vec{u} \in U$ & U is a subspace, $\alpha\vec{u} \in U$.
Similarly $\vec{u} \in V$ & V is a subspace so $\alpha\vec{u} \in V$. $\Big\} \Rightarrow \alpha\vec{u} \in U \cap V$
$\therefore$ $U \cap V$ is a subspace of W.

b) $U + V = \{u + v \mid u \in U \text{ and } v \in V\}$.
(1) Let $\vec{0} \in W$. Since U is a subspace $\vec{0} \in U$
Since V is a subspace $\vec{0} \in V$. note $\vec{0} = \vec{0} + \vec{0}$
So $\vec{0} \in U + V$.
(2) Let $u_1 + v_1, u_2 + v_2 \in U + V$.
Then $(u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2)$ by assoc & comm of + in W
Since U is a subspace $u_1 + u_2 \in U$ & similarly $v_1 + v_2 \in V$
So $(u_1 + u_2) + (v_1 + v_2) \in U + V$.
(3) Let $u + v \in U + V$ and $\alpha \in \mathbb{F}$
Since $\alpha(u + v) = \alpha u + \alpha v$ by dist in W
and $\alpha u \in U$ while $\alpha v \in V$ b/c both U & V are subspaces
$\alpha(u + v) \in U + V$.

c) Better statement: $U \cup V$ is a subspace if and only if $U \subset V$ or $V \subset U$.

$\Leftarrow$ Assume $U \subset V$. Then $U \cup V = V$ which is a subspace
If $V \subset U$. Then $U \cup V = U$ which is a subspace
$\Rightarrow$ Assume $U \cup V$ is a subspace, we need to show $U \subset V$ or $V \subset U$.
Assume $U \not\subset V$, we will show $V \subset U$.
Since $U \not\subset V$, there exists $\bar{u} \in U$ with the property $\bar{u} \notin V$.
Let v be an arbitrary element of V.
Note $v + \bar{u} \in U \cup V$ b/c our assumption that $U \cup V$ is a subspace.
If $v + \vec{u} \in U$ then $v + \vec{u} - \vec{u} \in U$ by closure of $U \Rightarrow v \in U$.
If $v + \vec{u} \in V$ then $v + \vec{u} - v \in V$ by closure of $V \Rightarrow \vec{u} \in V$ ※
Thus $\forall v \in V$, $v + \vec{u} \in U \Rightarrow v \in U \Rightarrow V \subset U$.

A similar argument shows if $V \not\subset U$ then $U \subset V$ //

18) $V + W = \{(a,b,c) + (x,y,0) \mid a = b = c \in \mathbb{R}\ \&\ x,y \in \mathbb{R}\}$
$= \{(a+x, a+y, a) \mid a,x,y \in \mathbb{R}\}$

- Given an arbitrary element $(\alpha,\beta,\gamma) \in \mathbb{R}^3$ we can set $a = \gamma$, $x = \alpha - \gamma$, and $y = \beta - \gamma$.
Then $(a+x, a+y, a) = (\gamma + \alpha - \gamma, \gamma + \beta - \gamma, \gamma) = (\alpha, \beta, \gamma)$
$\Rightarrow (\alpha,\beta,\gamma) \in V+W \Rightarrow \mathbb{R}^3 \subset V+W$
Since $V + W \subset \mathbb{R}^3 \Rightarrow \mathbb{R}^3 = V+W$

- Let $(\alpha,\beta,\gamma) \in V \cap W$.
Since $(\alpha,\beta,\gamma) \in V \Rightarrow \exists\ a \in \mathbb{R}$ so that $(\alpha,\beta,\gamma) = (a,a,a)$.
$\Rightarrow \alpha = a = \beta = \gamma$.
Since $(\alpha,\beta,\gamma) \in W \Rightarrow \exists\ x,y \in \mathbb{R}$ s.t. $(\alpha,\beta,\gamma) = (x,y,0)$
$\Rightarrow \gamma = 0$
Thus the first condition implies $\alpha = \beta = \gamma = 0$
$\Rightarrow (\alpha,\beta,\gamma) = (0,0,0)$.

19) Prove $V = \mathbb{R}^{n,n} = Mat_{n \times n}(\mathbb{R})$, then all diagonal matrices form a subspace.

Let D be the set of diagonal matrices, that is

$$\left\{ \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & \ddots \; a_n \end{bmatrix} \middle| \; a_i \in \mathbb{R} \text{ for } i = 1, \ldots n \right\}$$

We verify the three conditions.

(1) Let $a_i = 0$ for $i = 1, \ldots n$

then we have the zero matrix in D.

(2) Let $\begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{bmatrix}, \begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots \; b_n \end{bmatrix} \in D$

$$\begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{bmatrix} + \begin{bmatrix} b_1 & & 0 \\ & b_2 & \\ 0 & & \ddots \; b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 & & 0 \\ & a_2+b_2 & \\ 0 & & \ddots \; a_n+b_n \end{bmatrix} \in D. \checkmark$$

(3) Let $\alpha \in \mathbb{R}$ then

$$\alpha \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{bmatrix} = \begin{bmatrix} \alpha a_1 & & 0 \\ & \alpha a_2 & \\ 0 & & \ddots \; \alpha a_n \end{bmatrix} \in D \checkmark$$

20) $\text{vec} : Mat_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longmapsto \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

a) one-to-one correspondence (they mean 1-1 & onto)

Assume $\text{vec}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \text{vec}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix}$$

so $b_{ij} = a_{ij} \; \forall i,j$

$\Rightarrow$ our 2×2 matrices were the same to begin with

Note also given any $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$

that we can construct a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix} \in Mat_{2\times 2}(\mathbb{R})$

with the property $\text{vec}\begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ $\Rightarrow$ vec is onto $\mathbb{R}^4$.

b) To see if vec is linear

let $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in Mat_{2\times 2}(\mathbb{R})$ $\alpha, \beta \in \mathbb{R}$

$$vec\left(\alpha\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \beta\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = vec\left(\begin{bmatrix} \alpha a_{11}+\beta b_{11} & \alpha a_{12}+\beta b_{12} \\ \alpha a_{21}+\beta b_{21} & \alpha a_{22}+\beta b_{22} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \alpha a_{11}+\beta b_{11} \\ \alpha a_{21}+\beta b_{21} \\ \alpha a_{12}+\beta b_{12} \\ \alpha a_{22}+\beta b_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} \\ \alpha a_{21} \\ \alpha a_{12} \\ \alpha a_{22} \end{bmatrix} + \begin{bmatrix} \beta b_{11} \\ \beta b_{21} \\ \beta b_{12} \\ \beta b_{22} \end{bmatrix}$$

$$= \alpha\begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix} + \beta\begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix}$$

$$= \alpha\, vec\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) + \beta\, vec\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right)$$

so linear ?