§3.1 # ~~6~~, ~~7~~, ~~8~~, ~~11c,e~~, ~~14~~, ~~15~~, ~~16~~, ~~19~~, 23

6) $V = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is cont } \& \ f(0) = 0 \}$ w/ standard function + & scalar mult

note $g(x) = 0 \in V$ so V is nonempty.

Let $f, g, h \in V$ and $a, b \in \mathbb{R}$ be scalars.

(1) Math 251 $\Rightarrow$ $f+g$ is cont

Note $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$ so $f+g \in V$.

(2) Commutivity of $\mathbb{R}$ $\Rightarrow$ $f+g = g+f$

(3) Associativity of $\mathbb{R}$ $\Rightarrow$ $f + (g+h) = (f+g) + h$

(4) Consider $\vec{0} : [0,1] \to \mathbb{R}$ defined by $\vec{0}(x) = 0 \ \forall x$

note $\vec{0} \in V$ and $f + \vec{0} = f = \vec{0} + f$

(5) Given $f \in V$, consider $-f : [0,1] \to \mathbb{R}$ defined by $(-f)(x) = -(f(x))$, note $-f(0) = 0$ so $-f \in V$ &

(6) Math 251 $\Rightarrow$ af is cont

Note $(af)(0) = af(0) = a \cdot 0 = 0$ so $af \in V$

(7) & (8) $a(f+g)(x) = a(f(x) + g(x)) = af(x) + ag(x) = (af + ag)(x)$

$\Rightarrow a(f+g) = af + ag$

$(a+b)f(x) = af(x) + bf(x) = (af + bf)(x)$

$\Rightarrow (a+b)f = af + bf$

(9) $a(bf)(x) = a(bf(x)) = abf(x) = (ab)f(x)$

$\Rightarrow a(bf) = (ab)f$

(10) Consider $1 \in \mathbb{R}$.

<u>Is</u> a vector space

7) $V = \{ ax^2 + bx + c \mid a \neq 0 \}$ with standard function + & scalar mult

Closure problem: note $2x^2, -2x^2 \in V$

but $2x^2 + (-2x^2) = 0 \notin V$.

is <u>not</u> a vector space

8) $V = \{f : [0,1] \to \mathbb{R} \mid f \text{ is cont } \& \ f(0) = f(1)\}$.

note $g(x) = 0 \ \forall x$ is cont & $g(0) = g(0)$ so $g \in V$ & V is nonempty

let $f, g, h \in V$ and $a, b \in \mathbb{R}$.

(1) Math 251 $\Rightarrow$ $f+g$ is cont

$(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)(1)$ so $f+g \in V$

(2) By commutivity of $\mathbb{R}$ $f+g = g+f$

(3) By associativity of $\mathbb{R}$ $(f+g)+h = f+(g+h)$

(4) Consider $\vec{0} : [0,1] \to \mathbb{R}$ defined by $\vec{0}(x) = 0$.

note $f + \vec{0} = f = \vec{0} + f$

(5) Consider $-f : [0,1] \to \mathbb{R}$ defined by $(-f)(x) = -f(x)$.

Math 251 $\Rightarrow$ $-f$ is cont and $(-f)(0) = -f(0) = -f(1) = (-f)(1)$

so $-f \in V$

(6) Math 251 $\Rightarrow$ af is cont.

$(af)(0) = af(0) = af(1) = (af)(1)$ $\Rightarrow$ $af \in V$

(7) & (8) $a(f+g)(x) = a(f(x) + g(x)) = af(x) + ag(x) = (af + ag)(x)$

$\Rightarrow a(f+g) = af + ag$

$(a+b)f(x) = af(x) + bf(x) = (af + bf)(x)$

$\Rightarrow (a+b)f = af + bf$

(9) $a(bf)(x) = a(bf(x)) = (ab)f(x)$

$\Rightarrow a(bf) = (ab)f$

(10) Consider $1 \in \mathbb{R}$

11) let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$

T/F $\rightarrow$ c)

$$T\left(\alpha\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \beta\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha x + \beta a \\ \alpha y + \beta b \\ \alpha z + \beta c \end{bmatrix}\right) = \begin{bmatrix} \alpha y + \beta b \\ \alpha y + \beta b \end{bmatrix} \checkmark$$

$$\alpha T\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \beta T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha\begin{bmatrix} y \\ y \end{bmatrix} + \beta\begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} \alpha y + \beta b \\ \alpha y + \beta b \end{bmatrix}$$

T is linear

T is a 2×3 matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

range $\neq$ image b/c $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ but not in image

d) $T\left(\alpha\begin{bmatrix}x\\y\\z\end{bmatrix}+\beta\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=T\left(\begin{bmatrix}\alpha x+\beta a\\\alpha y+\beta b\\\alpha z+\beta c\end{bmatrix}\right)=(\alpha x+\beta a)\begin{bmatrix}0\\\alpha y+\beta b\end{bmatrix}=\begin{bmatrix}0\\(\alpha x+\beta a)(\alpha y+\beta b)\end{bmatrix}$

$\alpha T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)+\beta T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=\alpha x\begin{bmatrix}0\\y\end{bmatrix}+\beta a\begin{bmatrix}0\\b\end{bmatrix}=\begin{bmatrix}0\\\alpha xy+\beta ab\end{bmatrix}$

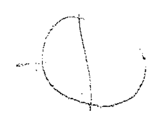
but $(\alpha x+\beta a)(\alpha y+\beta b)$ may not equal $\alpha xy+\beta ab$

So

~~ex~~ $T\left(\begin{bmatrix}1\\2\\0\end{bmatrix}+\begin{bmatrix}2\\1\\0\end{bmatrix}\right)=T\left(\begin{bmatrix}3\\3\\0\end{bmatrix}\right)=3\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}0\\9\end{bmatrix}\neq\begin{bmatrix}0\\4\end{bmatrix}=1\begin{bmatrix}0\\2\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}=T\left(\begin{bmatrix}1\\2\\0\end{bmatrix}\right)+T\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right)$

Not a linear operator

e) ~~ex~~ $T\left(\begin{bmatrix}0\\\pi/2\\\pi/2\end{bmatrix}+\begin{bmatrix}0\\\pi/2\\\pi/2\end{bmatrix}\right)=T\left(\begin{bmatrix}0\\\pi\\\pi\end{bmatrix}\right)=\begin{bmatrix}0\\-1\end{bmatrix}$



$\neq$

$T\left(\begin{bmatrix}0\\\pi/2\\\pi/2\end{bmatrix}\right)+T\left(\begin{bmatrix}0\\\pi/2\\\pi/2\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$

14) let $\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix},\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\in\mathbb{R}^{2,2}$ and $\alpha,\beta\in\mathbb{R}$

a) $T\left(\alpha\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}+\beta\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\right)=\alpha a_{22}+\beta b_{22}=\alpha T\left(\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\right)+\beta T\left(\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\right)$

linear

b) $T\left(\alpha\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}+\beta\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\right)=T\left(\begin{bmatrix}\alpha a_{11}+\beta b_{11}&\alpha a_{12}+\beta b_{12}\\\alpha a_{21}+\beta b_{21}&\alpha a_{22}+\beta b_{22}\end{bmatrix}\right)=\begin{bmatrix}\alpha a_{22}+\beta b_{22}&-(\alpha a_{12}+\beta b_{12})\\-(\alpha a_{21}+\beta b_{21})&\alpha a_{11}+\beta b_{11}\end{bmatrix}$

$=$

$\alpha T\left(\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\right)+\beta T\left(\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\right)=\alpha\begin{bmatrix}a_{22}&-a_{12}\\-a_{21}&a_{11}\end{bmatrix}+\beta\begin{bmatrix}b_{22}&-b_{12}\\-b_{21}&b_{11}\end{bmatrix}$

Started 4) (+1)
got 2 (+1)
got 3 (+1)

linear — This should remind you 2×2 inverses..!

c) not linear $T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)\neq T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$

$0=0+0$ $\qquad =1$

d) not linear

$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)+T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}\neq\begin{bmatrix}1\\0\end{bmatrix}=T\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$

let $\vec{x}, \vec{y} \in V$ & α, β be scalars

15) linear ✓ $\quad id(\alpha\vec{x} + \beta\vec{y}) = \alpha\vec{x} + \beta\vec{y} = \alpha\, id(\vec{x}) + \beta\, id(\vec{y})$

invertible ✓ $\quad$ use identity map for the inverse

(+1) 16) linear ✓ $\quad O_{UV}(\alpha\vec{x} + \beta\vec{y}) = 0 = 0 + 0 = \alpha O_{UV}(\vec{x}) + \beta O_{UV}(\vec{y})$

(+1) invertible? $\quad$ nope

T/F → 19) Prove $c\vec{0} = \vec{0}$ using only the 10 prop. of an abstract v.s.

Note $c\vec{0} = c(\vec{0} + \vec{0})$ $\quad$ by prop of $\vec{0}$,

$= c\vec{0} + c\vec{0}$ $\quad$ by dist prop.

Since inverses exist, $-(c\vec{0}) \in V$, so we apply this to both sides of the above to find $c\vec{0} + -(c\vec{0}) = (c\vec{0} + c\vec{0}) + -(c\vec{0})$

By def. of inverses & assoc $\Rightarrow$ $\vec{0} = c\vec{0}$. //

23) Let $(u_i, v_i) \in U \times V$ for $i = 1,2,3$ & α, β be scalars

Note since U & V are both nonempty, $U \times V$ is nonempty.

(1) The closure of $+$ of U & V $\Rightarrow$ $(u_1, v_1) + (u_2, v_2) \in U \times V$

(2) The commutivity of U & V $\Rightarrow$ $(u_1, v_1) + (u_2, v_2) = (u_2, v_2) + (u_1, v_1)$

(3) The assoc. of $U + V$ $\Rightarrow$ $(u_1, v_1) + ((u_2, v_2) + (u_3, v_3)) = ((u_1, v_1) + (u_2, v_2)) + (u_3, v_3)$

(4) Since U & V are v.s. there exist $\vec{0}_U \in U$ & $\vec{0}_V \in V$ that are the zero vectors. Note $(\vec{0}_U, \vec{0}_V)$ does the trick.

(5) Since U & V are v.s. $\exists$ $-\vec{u}_1 \in U$ and $-\vec{v}_1 \in V$ $\ni$ $\vec{u}_1 + -\vec{u}_1 = \vec{0}_U$ & $\vec{v}_1 + -\vec{v}_1 = \vec{0}_V$ so Given $(\vec{u}_1, \vec{v}_1) \in U \times V$ the element $(-\vec{u}_1, -\vec{v}_1)$ is the inverse

(6) The closure of $\cdot$ with respect to the same scalars $\Rightarrow$ $c(u_1, v_1) \in U \times V$

(7) & (8) The dist prop of U & V + the definition of $\cdot$ a scalar action on both coordinates $\Rightarrow$ $U \times V$ satisfies dist

(9). Assoc holds b/c both U & V are v.s. & the action is what it is

(10) We know $\exists$ $1_U \in \mathbb{F}$ & $1_V \in \mathbb{F}$ that give a monoidal structure on U & V resp. (We'll show that $1_U = 1 = 1_V$ where 1 is your mult id. in $\mathbb{F}$