Complex Numbers as matrices

We already think of complex numbers in a few different ways. In particular we can think of them as:

- $\{a + bi \mid a, b \in \mathbb{R}\}$ where $i^2 = -1$.
- $\{re^{i\theta} \mid r \in \mathbb{R}_+ \text{ and } 0 \leq \theta < 2\pi\}$

The first representation of a complex number is often called the standard form while the second is called polar form. The goal of this worksheet is to introduce another way of thinking about complex numbers as a certain subset of 2×2 matrices.

Define a function $F : \mathbb{C} \to \mathrm{Mat}_{2\times 2}(\mathbb{R}$ by $F(a + bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Here, $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ means 2 by 2 matrices with entries from $\mathbb{R}$. In particular, this means any scalars that will act on a matrix from $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ must also be from $\mathbb{R}$.

1. Find the following:
   (a) $F(3 - 4i)$
   (b) $F(1)$
   (c) $F(i)$
2. Write down what it means for F to be one-to-one and verify that F is one-to-one. One important consequence of this observation is that $\mathbb{C}$ can be identified with the image of the map F.
3. Show that $F(z + w) = F(z) + F(w)$ for any $z, w \in \mathbb{C}$. This means that F respects the addition of complex numbers.
4. Verify $[F(i)]^2 = F(i^2)$. (Recall that the multiplication on the left involves matrix multiplication.)
5. Show that $F(zw) = F(z)F(w)$ for any $z, w \in \mathbb{C}$. This means that F respects the multiplication of complex numbers.