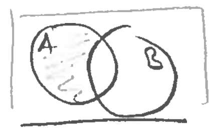
9:00
7:30
30min

FINAL — TCSS 321 — 75 — Fall 2012

1. [6] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true and briefly justify your answer. Otherwise, circle F and provide a counterexample or brief reasoning.

tera (+.5) swer (+1)

T F If A and B are sets then $A \setminus B = A \cap \overline{B}$ (+.5)



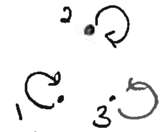
T F If $a|(bc)$ where a, b, c are integers greater than zero, then $a|b$ or $a|c$.

$4 \mid (2 \cdot 2)$ but $4 \nmid 2$ and $4 \nmid 2$

T F A relation on a set cannot be both symmetric and antisymmetric.

Let $A = \{1,2,3\}$ and

ie $\{(1,1),(2,2),(3,3)\}$



both symmetric and antisymmetric

Show your work for the following problems. The correct answer with no supporting work will receive NO credit. We use the logical symbols from the textbook unless otherwise specified.

2. (Quiz1 #3) Consider the following statement:
"There is a girl who eats curds and whey."

(a) 4[3] Express the statement above using predicates, quantifiers, and logical connectives. Be sure to *define* any terms your create.

sentence wrt x each (+2)

Let $C(x)$ be "x eats curds"
$W(x)$ be "x eats whey"

(+.5) Domain of x is girls.

$\exists x, \; C(x) \wedge W(x)$ (+.5) (+.5)

sense (+.5)

fix → (b) [2] Negate part (a) so that no negations are to the left of any quantifiers. Make use of DeMorgan's law

$\neg \exists x \; C(x) \wedge W(x)$ (+.5)

$\forall x \; \neg(C(x) \wedge W(x))$ (+.5)

$\forall x \; \neg C(x) \vee \neg W(x)$ (+1)

3. (proof con't wks #1) Read the following "Theorem". Determine (and justify)

   (a) [2] if the "Theorem" is true

   (b) [4] if the "Proof" is valid.

Theorem 1 *Let m, n, and p be integers. If $m+n$ and $n+p$ are even integers, then $m+p$ is an even integer.*

Proof 1 *Assume that $m+n$ and $n+p$ are even integers. We want to show that $m+p$ is an even integer.*

Since $m+n$ is an even integer, there exists integers a and b such that

$$m+n = 2a+2b.$$

Thus we know that $m = 2a$ and $n = 2b$.

Since $n+p$ is an even integer and $n = 2b$, we know that there exists and integer c such that

$$n+p = 2b+2c.$$

Thus we also know that $p = 2c$.

Now we can consider $m+p$ which equals $2a+2c = 2(a+c)$. Thus $m+p$ is even.

a) valid +.5

$m+n=2d$ & $n+p=2f$
Note $m+n+n+p=2d+2f$
$\Rightarrow$ $m+p=2d+2f-2n$
So $2 \mid (m+p) \Rightarrow m+p$ is even
+1

Start just .5

b) not valid +1

$m+n=2d$ for $d \in \mathbb{Z}$
we don't know that
m and n are even
+2.5

Start +.5

4. (§2.4 #17) Consider the sequence $\{a_n\}$ where $a_n = 2 \cdot n \cdot a_{n-1}$ where $a_0 = 1$.

   (a) [3] Write the first five terms of the sequence.

$a_0 = 1$
$a_1 = 2 \cdot 1 \cdot 1 = 2$ +.5
$a_2 = 2 \cdot 2 \cdot 2 = 2^2 \cdot 2! = 8$ +.5
$a_3 = 2 \cdot 3 \cdot 2^2 \cdot 2! = 2^3 \cdot 3! = 48$ +.5
$a_4 = 2 \cdot 4 \cdot 2^3 \cdot 3! = 2^4 \cdot 4! = 384$ +.5

start +.5
formula +.5

Fix

   (b) [4] Find a closed form (non-recursive definition) for a_n where $n \geq 1$

$a_n = \underbrace{2^n}_{+1} \; n!$ +1.5

Start +.5
indexing +1

5. Consider the recursive algorithm described below for the next few questions.

```
int RecursiveAlg(n)
Data: n: integer where n > 0
if n == 1 then
    return 1;
else
    return n· RecursvieAlg(n-1)
end
```

Algorithm 1: Recursive Algorithm

(a) [2] (Exam2 #4) Describe what the above algorithm does. Start (+.5)

returns $n!$ } (+1.5)

(b) [3] (recursive wks #5) Use induction to prove the recursively defined program computes what you claimed in part (a).

(+1) Base Case: Consider when n=1
RecursiveAlg(1) = 1 = 1! ✓

(+.5) Inductive:
(+.5) Assume RecursiveAlg(n-1) = (n-1)!
We need to show RecursiveAlg(n) = n!

(+.5) Notice
RecursiveAlg(n)
= n RecursiveAlg(n-1)
= n·(n-1)!
(+.5) by inductive assump
= n! ✓

(c) [4] (§3.1 #3) Write a program that is *not* recursive but completes the same computations done above.

```
Factorial(n)  # n: int and n > 0
answer = 1
for i = 1 to n
    answer = answer · i
return answer.
```

} initialize (+1)

} forloop/index (+1)
entry (+1)

style/sense (+1)

(d) [2] (§3.3 #) Give a big-Θ estimate for the number of products taken in your algorithm from part (c). Show your reasoning.

$\underbrace{n}_{(+1)}$, $\underbrace{\text{the for loop does exactly } n \text{ products}}_{(+1)}$

6. [4] (Exam2 #7) Find $112_3 \times 210_3$. Give your answer in decimal/base 10.

14 × 21 = 294

112_3 × 210_3 (+1) algorithm mult

1120 (+1) carries

100100

101220_3

$294 = 1\cdot3^5 + 0\cdot3^4 + 1\cdot3^3 + 2\cdot3^2 + 2\cdot3^1 + 0\cdot3^0$

conversion to decimal (+2)

7. [4] (§4.3 #31) The product of two integers is $2^4 \cdot 3^5 \cdot 7^4 \cdot 11$ and their greatest common divisor is $2^4 \cdot 3^2 \cdot 11$. What is the least common multiple of the two integers? Show your reasoning.

$a \cdot b = \gcd(a,b)\,\mathrm{lcm}(a,b)$ } (+1) stat (+.5)

lcm is max of exp 1.5

$2^4 \cdot 3^5 \cdot 7^4 \cdot 11 = 2^4 \cdot 3^2 \cdot 11 \cdot \mathrm{lcm}(a,b)$ } (+1) alg exp (+1)

$\mathrm{lcm}(a,b) = 3^3 \cdot 7^4$ } (+.5) $= 64827$

8. [4] (§4.4 #32) Solve the system of congruences ~~x ≡ 2 mod 6~~ and ~~x ≡ 3 mod 8~~.

$x \equiv 6 \bmod 8$

$x \equiv 3 \bmod 7$

(1) $x \equiv 6 \bmod 8$

(2) $x \equiv 3 \bmod 7$

Algorithm (+1)

$m = 56 \quad m_1 = 7 \quad m_2 = 8$

$a_1 = 6 \quad a_2 = 3$

$y_1 = 7 \quad y_2 = 1$ } inverse mod (+1.5)

inverse to 7 mod 8; inverse to 8 mod 7

$x = \underbrace{7\cdot6\cdot7 + 8\cdot3\cdot1}_{(+1)} \bmod 56$

$= 318 \bmod 56$

$= \underbrace{38 \bmod 56}_{(+1)}$

stat (+.5)

def of mod +1

CK: $38 \bmod 8 = 6$ ✓

$38 \bmod 7 = 3$ ✓

9. [5] Choose *ONE* of the following and support your conclusions. Clearly identify which of the two you are answering and what work you want to be considered for credit.

(a) (§5.1 #32) Prove 3 divides $n^3 + 2n$ whenever n is a positive integer.

(b) (§5.2 #3) Prove that a postage of n cents, for $n \geq 8$ can be formed using just 3-cent stamps and 5-cents stamps.

a) We'll use induction } (+1) / style

Base Case: Let $n=1$, notice $n^3+2n = 3$ and $3 \mid 3$ ✓ } (+1)

Induction: Assume $3 \mid [(n-1)^3 + 2(n-1)]$ we want to show $3 \mid (n^3+2n)$ } (+1)

Notice $(n-1)^3 + 2(n-1) = n^3 - 3n^2 + 5n - 3$

Since $3 \mid [(n-1)^3 + 2(n-1)]$

$3 \mid [n^3 + 2n - 3n^2 + 3n - 3]$

$\Rightarrow 3 \mid [n^3 + 2n]$ ✓

b) We'll use strong induction } (+1)

BC: $8 = 3¢ + 5¢ \quad 9 = 3¢ + 3¢ + 3¢$ } (+1)

$10 = 5¢ + 5¢$

Induction: Assume k can be formed using just 3¢ & 5¢ stamps when $k \leq n$. } (+1)

(+1) Notice $n+1 = (n-2) + 3$.

By induction assumption $n-2$ can be formed with x 3¢ stamps + y 5¢ stamps.

(+1) Thus $n+1 = x\cdot3¢ + y\,5¢ + 3¢$ //

4 10. [3] (functions wks #5) Define a function that is onto/surjective, but not one-to-one.

Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $x \mapsto x^2$ } defined (+2)

domain +.5
codomain +.5
rule +.5
sense +.5

onto (+1)
not one-to-one (+1)

11. [6] Choose *ONE* of the following and support your conclusions. Clearly identify which of the two you are answering and what work you want to be considered for credit.

(a) (11/14 lecture) Show if a, b, and m are integers such that $m \geq 2$ and $a \equiv b \mod m$, then the $\gcd(b, m)$ divides the $\gcd(a.m)$.

(b) (§9.1 #50) Let R and S be reflexive relations on a set A. Show that $R \oplus S$ is irreflexive.

a) We'll use a direct proof } (+1)

terms (+1)

Since $a \equiv b \mod m$
$\exists\, n \in \mathbb{Z} \ni a = b + nm$, } (+1)

By def $\gcd(a,m) \mid a$
and $\gcd(a,m) \mid m$ (+.5)
thus $\gcd(a,m) \mid nm$ prop (+1)
and $\gcd(a,m) \mid (a - nm)$. logic (+1)
So $\gcd(a,m) \mid b$.
Thus $\gcd(a,m)$ divides both b and m.

By def $\gcd(b,m)$ divides both b and m b/c it is the smallest to do so
thus
$\gcd(b,m) \mid \gcd(a,m)$.

logic (+.5)

b) We'll use a direct proof. } (+1)

(+1) Since R is reflexive $(a,a) \in R$ $\forall a \in A$ and b/c S is too, $(a,a) \in S$ $\forall a \in A$.

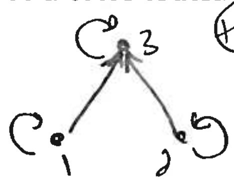
(+1) Recall $R \oplus S$ is the exclusive or and contains only elements from R that are not in S and elements from S not in R.
i.e. $R \oplus S = (R \setminus S) \cup (S \setminus R)$

(+1) Since $(a,a) \in R$ and $(a,a) \in S$ by def of $\oplus$ $(a,a) \notin R \oplus S$, $\forall a \in A$

Thus $R \oplus S$ is irreflexive.

quantifiers $\forall$ in right spots (+1)
logic (+1)

4 12. [~~3~~] (12/5 lecture) Define a set and a relation on that set that is a partial ordering but not a total ordering.



(+1.5) poset: reflexive, antisymmetric transitive

(+1) total ordering: 1 is not comparable to 2

start (+.5)

defined clearly (+1)

~~delegate~~ defines clearly +1

13. [6] Choose *ONE* of the following and support your conclusions. Clearly identify which of the two you are answering and what work you want to be considered for credit.

Start (+1)
logic (+2)
clear/sense (+1)
complete (+1)
answer (+1)

(a) (Exam1 #8) Suppose you are on an island that has two kinds of inhabitants, knights who always tell the truth, and their opposites, knaves who always lie. You encounter two people A and B. What are A and B if A says "I am a knave or B is a knight." and B says nothing.

(b) (Exam2 #6) Clearly describe an algorithm that finds all terms of a finite sequence of integers that are greater than the sum of all previous terms in the sequence. Start (+1)

a) There are 4 possibilities:

	A	B
1)	Knight	Knight
2)	Knight	Knave
3)	Knave	Knight
4)	Knave	Knave

$\boxed{\text{Possibility 1: is a possibility}}$

Possibility 2: <u>not</u> possible b/c A is a Knight but would be lying

Possibility 3: <u>not</u> possible if A is a Knave the negation of the statement would be "I am a Knight and B is a Knave" must be true, but it's not

Possibility 4: <u>not</u> possible. The negation of A's statement would still be false

b) GreaterThanSum ($a_1, a_2, \ldots a_n$: integers)

List of Terms = [a_1]
RunningSum = a_1 } initialize (+1.5)
for i = 2 to n index (+1)
 if a_i > RunningSum
  append a_i to ListofTerms
  RunningSum = a_i + RunningSum
return ListofTerms.

logic (+2)
style (+.5)