Key

EXAM 2 # TCSS 321 Winter 2013

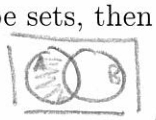
1. [6] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true and briefly justify your answer. Otherwise, circle F and provide a counterexample or brief reasoning.

   T F $\{2, -4\} \in \{8.3, -4, 7.2, 4, 2, 0\}$.

   False, the set on the right has no sets as elements.

   (+.5) start reasoning (+.5) reasoning (+1)

   T F Let A and B be sets, then $A \setminus B = A \cap \overline{B}$.

   (+.5) A-B  =  $A \cap \overline{B}$ $\Leftarrow \overline{B}$  def of set +.5

   start reasoning (+.5) reasoning (+1)

   T F Let A, B, C, and D be sets, then $(A \times B) \times (C \times D) = A \times (B \times C) \times D$.

   (+.5) the elements in $(A\times B)\times(C\times D)$ are of the form $((a,b),(c,d))$ whereas elements in $A\times(B\times C)\times D$ are of the form $((a,(b,c)),d)$.

   start reasoning (+.5) reasoning (+1)

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. [3] (HW3 §2.5 #3) Give an example of two uncountable sets A and B, such that $A \cap B$ is finite.

A is uncountable (+.5)
B is uncountable (+.5)
def/know intersection (+.5)
$A \cap B$ is finite (+1)
sense/notation (+.5)

ex
$A = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$
$B = \{x \in \mathbb{R} \mid -1 \le x \le 0\}$
note
$A \cap B = \{0\}$
which has only one element

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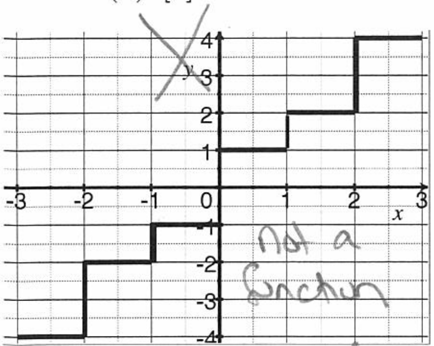
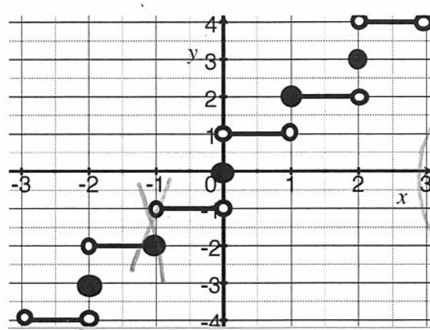
3. (HW4 §2.3 #3) Let $f(x) = \lfloor x \rfloor + \lceil x/2 \rceil$ be a map from $\mathbb{R}$ to $\mathbb{R}$.

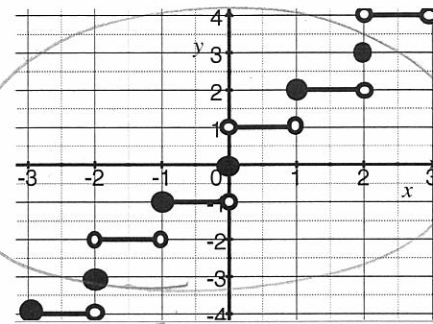
(a) [2] Evaluate $f(-2)$. $f(-2) = \underbrace{\lfloor -2 \rfloor + \lceil \frac{-2}{2} \rceil}_{+1} = -2 + -1 = \underbrace{-3}_{+1}$

(b) [2] Which of the following is a graph of f? Briefly justify yourself.



Not a function





justification: $f(-1) = \lfloor -1 \rfloor + \lceil -1/2 \rceil = -1 + \lceil -.5 \rceil = -1 + 0 = -1$

4. Each of the following give an exact count of operations taken in a worse case senerio for a certain algorithm. Identify their complexity (big-O for programmers or big-Θ for computer scientists) as you would to a colleague and justify yourself.

(a) [2] (§3.2 #5) $\dfrac{x^2+1}{x+1}$

(+.5) { Claim: $\dfrac{x^2+1}{x+1} \in O(x)$

start justify (+.5)
reasoning (+1)

My justification:

$$\lim_{x\to\infty} \frac{\frac{x^2+1}{x+1}}{x} = \lim_{x\to\infty} \frac{x^2+1}{x(x+1)} = \lim_{x\to\infty} \frac{x^2+1}{x^2+1} = 1 < \infty$$

other justifications to CS people: The numerator is dominated by x^2 & the denominator is dominated by x. The ratio & thus big O is x

other justifications.

(b) [2] (HW4 §3.2 #2) $\sqrt{n} + \log n$

(+.5) { Claim: $\sqrt{n} + \log n \in O(\sqrt{n})$

start justify (+.5)
reasoning (+1)

My justification: $\max(O(\sqrt{n}), O(\log n)) = O(\sqrt{n})$

notice that $\log n \in O(\sqrt{n})$ b/c

$$\lim_{n\to\infty} \frac{\log n}{\sqrt{n}} \underset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{(\ln 2) n} = \lim_{n\to\infty} \frac{2}{\ln 2 \sqrt{n}} = 0$$

(c) [2] (HW4 §3.2 #3) $1 + 2 + 3 + \ldots + n$

(+.5) { Claim: $1+2+3+\ldots+n \in O(n^2)$

start justify (+.5)
reasoning (+1)

My justification:

$$\begin{array}{r} 1+2+3+\ldots+n \\ +\ n+(n-1)+\ldots+1 \\ \hline \underbrace{(n+1)+(n+1)+\ldots+(n+1)}_{n \text{ times}} \end{array}$$

$\Rightarrow 2(1+2+3+\ldots+n) = n(n+1)$

$\Rightarrow 1+2+3+\ldots+n = \frac{1}{2} n(n+1)$

So $1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$ and $\dfrac{n(n+1)}{2} \in O(n^2)$

other justifications

$1+2+\ldots+n \le \underbrace{n+n+\ldots+n}_{n \text{ times}} = n \cdot n = n^2$

So $1+2+\ldots+n \in O(n^2)$

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5. (§2.4 #17) Consider the sequence $\{a\}$ defined recursively by $a_n = a_{n-1} + 2$ where $a_0 = 3$.

(a) [2] Find a_1 and a_2.

used recursive def (+.5)
used initial value (+.5)

$a_1 = a_0 + 2 = 3+2 = 5$ (+.5) $a_2 = a_1 + 2 = 5+2 = 7$ (+.5)

(b) [4] Find a (closed form) formula for a_n as a function of n (as opposed to using any terms previous to a_n in the formula).

Start (+1)

$a_0 = 3$
$a_1 = 3+2 = 3+2\cdot 1$
$a_2 = 3+2+2 = 3+2\cdot 2$
$a_3 = 3+2+2+2 = 3+3\cdot 2$
$a_4 = 3+2+2+2+2 = 3+4\cdot 2$

see a few terms (+1)

$a_n = \underbrace{3}_{(+1)} + \underbrace{n\cdot 2}_{(+1)}$

6. [3] (§2.4 #33) Compute $\displaystyle\sum_{i=0}^{2}\left(\sum_{j=0}^{3}(2i+3j)\right)$.

$i=0$: $(2\cdot0+3\cdot0)+(2\cdot0+3\cdot1)+(2\cdot0+3\cdot2)+(2\cdot0+3\cdot3) = 3+6+9 = 18$
$i=1$: $(2\cdot1+3\cdot0)+(2\cdot1+3\cdot1)+(2\cdot1+3\cdot2)+(2\cdot1+3\cdot3) = 2+5+8+11=26$
$+$ $i=2$: $(2\cdot2+3\cdot0)+(2\cdot2+3\cdot1)+(2\cdot2+3\cdot2)+(2\cdot2+3\cdot3) = 4+7+10+13=34$

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range of i (+1) range of j (+1) forms (+1)

7. [6] (matrix wks) Let $M = \begin{bmatrix} a & c \\ 0 & c \\ 0 & 0 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 & b \\ 0 & f & a \end{bmatrix}$, and $P = \begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$,

where a, b, c, and f are nonzero real numbers. Find the following if possible:

$M + P$

$$\begin{bmatrix} a & c \\ 0 & c \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a+1 & c \\ 0 & c+f \\ f & 0 \end{bmatrix}}_{(+2)}$$

$M^T + P$

$M^T = \begin{bmatrix} a & 0 & 0 \\ c & c & 0 \end{bmatrix}$ (+1)

M^T is a 2×3
P is a 3×2

(+1) The matrices are dif sizes and can't be added.

NP

$$\begin{bmatrix} 1 & 0 & b \\ 0 & f & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$$

get 2×2 (+.5)

$$\begin{bmatrix} 1+0+bf & 0+0+0 \\ 0+0+af & 0+f^2+0 \end{bmatrix}$$

(+1) $\begin{bmatrix} 1+bf & 0 \\ af & f^2 \end{bmatrix}$

algorithm (+.5)

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typo

8. (matrix wks #6) Consider the algorithm described in pseudocode below for the next three questions.

Data: A, B: $n \times n$ matrices where $n \geq 2$
for $i = 1$ *to* n **do**
  for $j = 1$ *to* n **do**
    $c_{ij} := 0$
    if $i \leq j$ **then**
      for $q := i$ *to* j **do**
        $c_{ij} := c_{ij} + a_{iq}b_{qj}$
      end
    end
  end
end
return C $\#C = [c_{ij}]$

Algorithm 1: Matrix Algorithm

(a) [4] (§3.1 #1) Use Algorithm 1 by hand on $A = \begin{bmatrix} a & b \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$. Clearly indicate your computations for each i and j.

+1

Start +.5
followed algorithm +3
got it +.5

$i=1$, $j=1$: $c_{11}=0$; if $i \le j$ ✓; $c_{11} = 0 + a_{11}b_{11}$ $\Rightarrow c_{11} = a \cdot 1$

$i=1$, $j=2$: $c_{12} = 0$; if $i \le j$ ✓; $c_{12} = 0 + a_{11}b_{12} + a_{12}b_{22}$ $\Rightarrow c_{12} = a3 + b2$

$i=2$, $j=1$: $c_{21} = 0$; if $i \le j$ ✗ $\Rightarrow c_{21} = 0$

$i=2$, $j=2$: $c_{22} = 0$; if $i \le j$ ✓; $c_{22} = 0 + a_{22}b_{22}$; $c_{22} = b2$

returns $\begin{bmatrix} a & 3a+2b \\ 0 & 2b \end{bmatrix}$ ie the product of the 2 upper Δ matrices.

(b) [2] Describe what the algorithm is doing as you would to a colleague during lunch.

start +.5
matrix mult +1 +.5

The algorithm can multiply two upper Δ matrices
(faster than using the standard definition of matrix multiplication)

(c) [4] (§3.3 #3) Give a big-O estimate for the number of multiplications used in Algorithm 1 as a function of n. Show your reasoning. Start +.5 got it +.5

for an $n \times n$ upper Δ matrix
first row needs $1+2+3+\dots+n$
2nd row needs $1+2+\dots+(n-1)$
3rd row needs $1+2+\dots(n-2)$
n^{th} row needs 1

notice the total # mult is less than $n(1+2+3+\dots+n)$. From #4c
$= n\left(\frac{n(n+1)}{2}\right)$ so this is $O(n^3)$

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for a 3x3 upper Δ matrix $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$

c_{11} would take 1 multiplication
c_{12} " " 2 "
c_{13} " " 3 "
c_{21} " " 0 "
c_{22} " " 1 "
c_{23} " " 2 "
c_{31} " " 0
c_{32} " " 0
c_{33} " " 1 "

partial points
nested for loops +1
limits of nested for loops +2
consider ex +1

9. [6] Choose *ONE* of the following and support your conclusions. Clearly identify which of the two you are answering and what work you want to be considered for credit. Let A and B be sets.

   (a) (Quiz2 #2) Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

   (b) (HW3 #3) Prove if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

(a) We will show two things: start (+.5)

(i) $P(A) \subseteq P(B) \to A \subseteq B$ structure (+1)

(ii) $A \subseteq B \longrightarrow P(A) \subseteq P(B)$ intro/assume (+.5)

i) We will show this conditional statement by using the logical equivalence to the contrapositive. That is, we assume $A \not\subseteq B$ and will show $P(A) \not\subseteq P(B)$. (+.5)

(+.5) notation Since $A \not\subseteq B$, there exists an element $x \in A$ that is not in B. (+.5) sense

(+.5) logic Consider the set $\{x\}$. Note $\{x\} \in P(A)$ but since $x \notin B$ $\{x\} \notin P(B)$. Thus $P(A) \not\subseteq P(B)$.

ii) We assume $A \subseteq B$ and will show $P(A) \subseteq P(B)$. To show $P(A) \subseteq P(B)$ we assume a set $X \in P(A)$ and will show $X \in P(B)$. (+.5)

Since $X \in P(A)$, X is a subset of A. Thus, for all $z \in X$, $z \in A$. Since $z \in A \subseteq B$ we know all $z \in X$ are also in B. Thus X is a subset of $B \Rightarrow X \in P(B)$.

notation (+.5) sense (+.5) logic (+.5)

Thus we've shown both directions of the biconditional statement.

(b) ~~Assume~~ We will show this conditional statement by proving the contrapositive. That is, we assume $A \neq B$ then will show $P(A) \neq P(B)$. (+1)

Since $A \neq B$ we can assume, without loss of generality, $\exists$ $x \in A$ that is not an element of B.

Consider the set $\{x\}$. Since $x \in A$ we know $\{x\} \in P(A)$. However, since $x \notin B$, $\{x\} \notin P(B)$.

Thus we've identified an element in $P(A)$ not in $P(B) \Rightarrow P(A) \neq P(B)$, finishing the proof.

start (+.5)
structure (+1)
intro/assume (+.5)

notation (+1)
sense (+1)
logic (+1)

def of powerset +.5

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[illegible]

[illegible]

[illegible]

$$\begin{array}{r} 2 \\ 19 \\ 25 \\ 6 \\ \hline 50 \end{array}$$