Typos pg 4
Swap out #10a

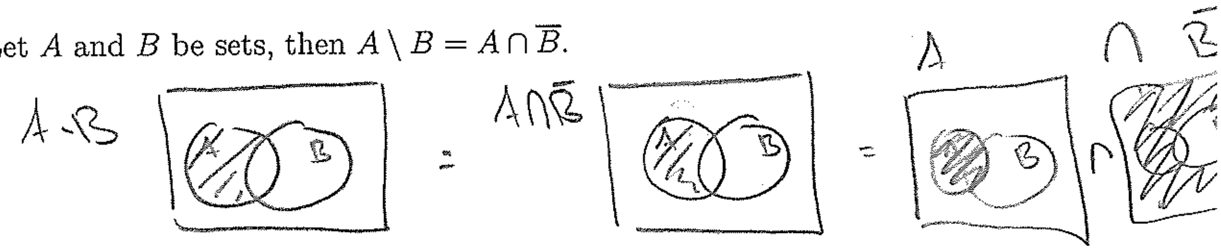
EXAM 2 # TCSS 321 PRACTICE

1. TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true and briefly justify your answer. Otherwise, circle F and provide a counterexample or brief reasoning.

T **(F)** $\{2, -4\} \in \{8.3, -4, 7.2, 4, 2, 0\}$.

False, the set on the right has no sets as elements.

(T) F Let A and B be sets, then $A \setminus B = A \cap \overline{B}$.



T **(F)** Let A, B, C, and D be sets, then $(A \times B) \times (C \times D) = A \times (B \times C) \times D$.

The elements in $(A \times B) \times (C \times D)$ are of the form $((a,b),(c,d))$ whereas elements in $A \times (B \times C) \times D$ are of the form $(a,(b,c),d)$

T **(F)** The intersection of two uncountable sets cannot be finite.

Could be finite? for example
$A = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ and
$B = \{x \in \mathbb{R} \mid -1 \le x \le 0\}$ while $A \cap B = \{0\}$

(T) F Let S be the set that contains a set x if the set x does not belong to itself. Then S is both a member and not a member of S

Russel's Paradox?
If S does not belong to itself, then by definition S is a member. But it was said that "S does not belong to itself," so it can't be a member
If S does belong to itself then by definition, S is not a member. But it was said that "S does belong to itself" so it has to be a member?

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Let $f(x) = \lfloor x \rfloor + \lceil x/2 \rceil$ be a map from $\mathbb{R}$ to $\mathbb{R}$.

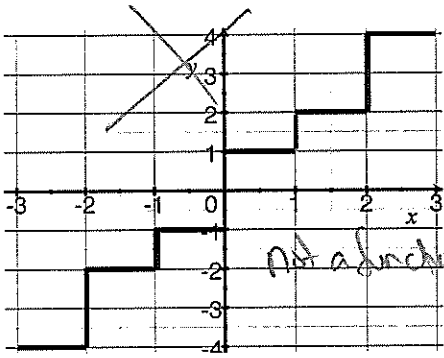
   (a) Evaluate $f(-2)$.

   $\lfloor -2 \rfloor + \lceil \frac{-2}{2} \rceil = -2 + -1 = -3$

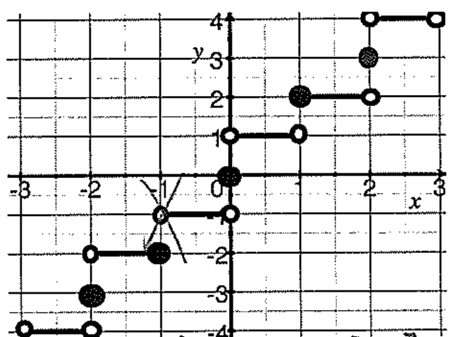
   (b) What is the range of f?

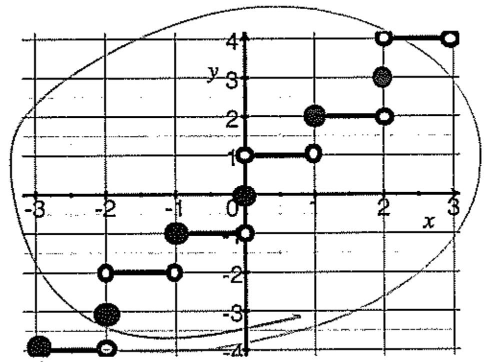
   $\mathbb{Z}$

   (c) Which of the following is a graph of f? Briefly justify yourself.



not a function





$f(-1) = \lfloor -1 \rfloor + \lceil -\frac{1}{2} \rceil = -1 + 0 = -1$

3. Each of the following identify their complexity (big-O for programmers or big-Θ for computer scientists). Justify your conclusions as you would to a colleague.

   (a) $\dfrac{x^2+1}{x+1}$ is $O(x)$

   b/c $\lim_{x\to\infty} \dfrac{\frac{x^2+1}{x+1}}{x} = \lim_{x\to\infty} \dfrac{x^2+1}{x(x+1)} = \lim_{x\to\infty} \dfrac{x^2+1}{x^2+x} \underset{L'H}{=} \lim_{x\to\infty} \dfrac{2x}{2x+1} \underset{L'H}{=} \lim_{x\to\infty} \dfrac{2}{2} = 1$

   $1 < \infty$

   (b) $\sqrt{n} + \log n$ is $O(\sqrt{n})$

   b/c $\lim_{n\to\infty} \dfrac{\sqrt{n} + \log n}{\sqrt{n}} = \lim_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt{n}} + \dfrac{\log n}{\sqrt{n}} = \lim_{n\to\infty} 1 + \dfrac{\log n}{n^{\frac{1}{2}}} \underset{L'H}{=} 1 + \lim_{n\to\infty} \dfrac{\frac{1}{n(\ln 2)}}{\frac{1}{2}n^{-\frac{1}{2}}}$

   $= 1 + \lim_{n\to\infty} \dfrac{1}{n(\ln 2)} \div \dfrac{1}{2n^{\frac{1}{2}}} = 1 + \lim_{n\to\infty} \dfrac{2n^{\frac{1}{2}}}{n \ln 2} = 1 + \lim_{n\to\infty} \dfrac{2}{\sqrt{n}\ln 2} = 1 + 0 < \infty$

   (c) $1^3 + 2^3 + 3^3 + \ldots + n^3$

   is $O(n^4)$

   b/c $1^3 + 2^3 + 3^3 + \ldots + n^3 \leq \overbrace{n^3 + n^3 + n^3 + \ldots + n^3}^{n} = n(n^3) = n^4$

4. Consider the sequence $\{a\}$ defined recursively by $a_n = a_{n-1} + 2$ where $a_0 = 3$.

(a) Find a_1 and a_2.

$a_1 = a_0 + 2 = 3+2 = 5$ $a_2 = a_1 + 2 = 5+2 = 7$

(b) Find a (closed form) formula for a_n as a function of n (as opposed to using any terms previous to a_n in the formula).

$a_0 = 3$
$a_1 = 3+2 = 3+2\cdot1$
$a_2 = 3+2+2 = 3+2\cdot2$
$a_3 = 3+2+2+2 = 3+3\cdot2$
$a_4 = 3+2+2+2+2 = 3+4\cdot2$

$a_n = 3 + 2n$

5. Compute $\sum_{i=0}^{3}\left(\sum_{j=0}^{4}(2i+3j)\right)$.

$i=0$ $(2(0)+3(0)) + (2(0)+3\cdot1) + (2(0)+3\cdot2) + (2(0)+3\cdot3) + (2(0)+3\cdot4)$
$i=1$ $+ (2(1)+3(0)) + (2(1)+3\cdot1) + (2(1)+3\cdot2) + (2(1)+3\cdot3) + (2(1)+3\cdot4)$
$i=2$ $+ (2(2)+3(0)) + (2(2)+3\cdot1) + (2(2)+3\cdot2) + (2\cdot2+3\cdot3) + (2\cdot2+3\cdot4)$
$i=3$ $+ (2\cdot3+3\cdot0) + (2\cdot3+3\cdot1) + (2\cdot3+3\cdot2) + (2\cdot3+3\cdot3) + (2\cdot3+3\cdot4)$

$\Rightarrow$

6. Let $M = \begin{bmatrix} a & c \\ 0 & c \\ 0 & 0 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 & b \\ 0 & f & a \end{bmatrix}$, and $P = \begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$,

where a, b, c, and f are nonzero real numbers. Find the following if possible:

$M + P$

$$\begin{bmatrix} a & c \\ 0 & c \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$$

$$\begin{bmatrix} a+1 & c \\ 0 & c+f \\ f & 0 \end{bmatrix}$$

$M^T + P$

$$M^T = \begin{bmatrix} a & 0 & 0 \\ c & c & 0 \end{bmatrix}$$

(2×3)

P is a 3×2

The matrices are different sizes & so can't be added

NP

$$\begin{bmatrix} 1 & 0 & b \\ 0 & f & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$$

(2×3) (3×2)

will get a 2×2

$$\begin{bmatrix} 1+0+bf & 0+0+0 \\ 0+0+af & 0+f^2+0 \end{bmatrix}$$

$$\begin{bmatrix} 1+bf & 0 \\ af & f^2 \end{bmatrix}$$

typo

7. Consider the algorithm described in pseudodocode below for the next three questions.

> **Data**: A, B: $n \times n$ matrices where $n \geq 2$
> **for** $i = 1$ *to* n **do**
>   **for** $j = 1$ *to* n **do**
> $c_{ij} := 0$
>   **if** $i \leq j$ **then**
>   **for** $q := i$ *to* j **do**
> $c_{ij} := c_{ij} + a_{iq}b_{qj}$
>   **end**
>   **end**
>   **end**
> **end**
> return C $\#C = [c_{ij}]$

Algorithm 1: Matrix Algorithm

(a) Use Algorithm 1 by hand on $A = \begin{bmatrix} a & b \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$.
Clearly indicate your computations for each i and j.

$i=1$, $j=1$ $c_{11}=0$ if $i \leq j$ ✓ $c_{11} = 0 + a_{11}b_{11}$ $\Rightarrow c_{11} = a \cdot 1$	$i=1$, $j=2$ $c_{12}=0$ if $i \leq j$ ✓ $c_{12} = 0 + a_{11}b_{12} + a_{12}b_{22}$ $\Rightarrow c_{12} = a3 + b2$	$i=2$, $j=1$ $c_{21}=0$ if $i \leq j$ ✗ $\Rightarrow c_{21} = 0$	$i=2$, $j=2$ $c_{22}=0$ if $i \leq j$ $c_{22} = 0 + a_{22}b_{22}$ $\Rightarrow c_{22} = 2b$

returns $\begin{bmatrix} a & 3a+2b \\ 0 & 2b \end{bmatrix}$

(b) Describe what the algorithm is doing as you would to a colleague during lunch.

The algorithm can multiply two upper Δ matrices (faster than using the standard definition of matrix mult)

(c) Give a big-O estimate for the number of multiplications used in Algorithm 1 as a function of n. Show your reasoning.

for an $n \times n$ up
first row needs $1+2+3+\dots+n$
2nd row needs $1+2+\dots+(n-1)$
3rd row needs $1+2+\dots+(n-2)$
nth row needs 1

notice the total # of mult $< n(1+2+\dots+n)$
$\Rightarrow$ total # of mult $< n\left(\frac{n(n+1)}{2}\right)$ 4
so is $O(n^3)$

for a 3×3 upper Δ matrix
c_{11} needs 1 mult
c_{12} needs 2 mult
c_{13} needs 3 mult
c_{21} needs 0 mult
c_{22} needs 1 mult
c_{23} needs 2 mult
c_{31} needs 0 mult
c_{32} needs 0 mult
c_{33} needs 1 mult

8. Create and describe an algorithm for detecting if a string can be reordered into a palindrome.

I'd like to track to make sure each letter shows up an even number of times. Note there can be at most one letter that appears an odd number of times that could be put in the middle

```
Input : string
Create : oddcount := 0
         ABCcount := [0][0] # double array
for i=0 to len(string)
    usedflag := 0
    for j=0 to len(ABCcount[0])-1
        if string[i] = ABCcount[j,0]
            (ABCcount[j][1] +1) % 2
            used flag = 1
    if used flag = 0
        append string[i] to ABCcount
for j=0 to len(ABCcount[0]-1)
    oddcount = oddcount + ABCcount[j,1
if oddcount > 1   return NO
```

9. Choose *ONE* of the following and support your conclusions. Clearly identify which of the two you are answering and what work you want to be considered for credit. Let A and B be sets.

(a) Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

(b) Prove if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

(a)

$\Leftarrow$ Assume $A \subseteq B$, we will show that $P(A) \subseteq P(B)$. Let $x \in P(A)$, then x is a subset of A. So $x \subseteq A \subseteq B$ thus $x \subseteq B \Rightarrow x \in P(B)$.

$\Rightarrow$ We will show if $P(A) \subseteq P(B)$ then $A \subseteq B$ by showing the contrapositive. That is, assume $A \nsubseteq B$ and we will show $P(A) \nsubseteq P(B)$. Since $A \nsubseteq B$, there exists $a \in A$ that is also not in B. Consider $\{a\}$; Note $\{a\} \subseteq A$ so $\{a\} \in P(A)$ but $\{a\} \nsubseteq B$ so $\{a\} \notin P(B)$. Thus $P(A) \nsubseteq P(B)$.

(b) We will show the contrapositive, that is assume $A \neq B$, we will show $P(A) \neq P(B)$.

Since $A \neq B$ there exists an element in one set that is not in the other. Without loss of generality assume $x \in A$ but $x \notin B$.

Notice $\{x\} \subseteq A$ so $\{x\} \in P(A)$. But $\{x\} \nsubseteq B$ so $\{x\} \notin P(B)$. Thus $P(A) \neq P(B)$

10. Choose *ONE* of the following and support your conclusions. Clearly identify which of the two you are answering and what work you want to be considered for credit. Let A and B be sets.

Strong induction...

(a) Consider a rectangular checkerboard that you'd like to divide into its constituent squares. You can either make (complete) cuts along the vertical lines or horizontal lines separating the squares. Determine if there is a formula to determine how many cuts you must make to break the checkerboard into the n separate squares. If there is a formula, justify it. If there is no formula, prove that it does not exist.

(b) For all positive integers n, $\sum_{i=1}^{n}(i^3) = \left(\frac{n(n+1)}{2}\right)^2$

(a)

$1n \Rightarrow n-1$

$2n \Rightarrow 1 + 2(n-1) = 2n - 1$

$3n \Rightarrow 2 + 3(n-1) = 3n - 1$

Claim: It will take $n-1$ cuts.

Pf: We will induct on the number of rows

Base Case: Consider a rectangular checkerboard with one row.
If there are n squares in one row then $n-1$ cuts are needed to cut the space between the pairs of squares

Induction: Consider a rectangle with $K+1$ rows and assume that we have the above formula for all rectangles with K rows.
Cut the bottom row off, notice we know have 2 rectangles, one with 1 row & one with K rows. Thus our total number of cuts will be
$1 +$ (cuts for 1 row rect) $+$ (cuts for K rows rect)
Let the original rectangle be $K+1$ by b so $(K+1)b = n$. The above then is

$$1 + (b-1) + (K \cdot b - 1)$$
$$= \cancel{1} + b - \cancel{1} + Kb - 1 = b(1+K) - 1$$
$$= n - 1 \quad //$$

(b) We will use induction?

Base case: Let $n=1$ notice $\sum_{i=1}^{1}(i)^3 = 1$ and $\left(\frac{1(1+1)}{2}\right)^2 = 1$

Induction: Assume $\sum_{i=1}^{n}(i^3) = \left(\frac{n(n+1)}{2}\right)^2$

we want to show that

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+1+1)}{2}\right)^2 \text{ or } \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

Consider

$$\sum_{i=1}^{n+1} i^3 = \underbrace{1^3 + 2^3 + \dots + n^3}_{} + (n+1)^3$$
$$= \sum_{i=1}^{n} i^3 + (n+1)^3 \text{ by the assumpt}$$
$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$
$$= \frac{n^2(n+1)^2}{4} + (n+1)^3$$
$$= (n+1)^2\left[\frac{n^2}{4} + (n+1)\right]$$
$$= (n+1)^2\left[\frac{n^2 + 4(n+1)}{4}\right]$$
$$= (n+1)^2 \frac{n^2+4n+4}{4}$$
$$= (n+1)^2 \frac{(n+2)^2}{2^2} = \left[\frac{(n+1)(n+2)}{2}\right]^2$$