2. The set of *full binary (and rooted) trees*, where a rooted tree consists of a set of vertices containing a distinguished vertex called the *root*, and eyes connecting these vertices, can be defined recursively by:

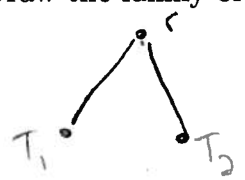
Basis Step: A single vertex r is a full binary (and rooted) tree.
Recursive Step: Suppose that T_1 and T_2 are disjoint full binary (and rooted) trees, there is a full binary (and rooted) tree, denoted $T_1 \cdot T_2$, consisting ~~for~~ of a root r together with edges connecting r to the root of the left subtree T_1 and r to the root of the right subtree T_2.

(a) Draw the basis step tree. 

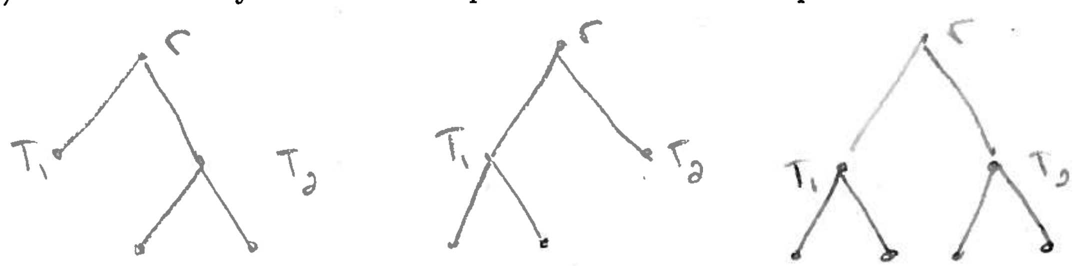
height: 0

(b) Draw the family of trees one step "above" the basis step tree.

height: 1



(c) Draw the family of trees two steps "above" the basis step tree.



(d) Let $h(T)$ be the height of a tree.
We define $h(\cdot) = 0$, and $h(T_1T_2) = 1 + \max\{h(T_1), h(T_2)\}$.
Find the height of the trees in (c). heights: 2 $\quad 1 + \max(T_1, T_2) = 1 + 1 = 2$

3. Let T be a binary tree, and $n(T)$ be the number of vertices in T. Use induction to prove if T is a full binary tree, then $n(T) \geq 2h(T) + 1$.

Base Case: Note $n(\cdot) = 1 \geq 2 \cdot 0 + 1 = 2h(\cdot) + 1$ ✓

Induction: We induct by using the recursive def of trees.
Let T be a full binary tree. By construction $\exists\ T_1 + T_2$ $\ni T = T_1 \cdot T_2$ and by our inductive assumption $n(T_i) \geq 2h(T_i) + 1$

Notice $n(T) = 1 + n(T_1) + n(T_2)$ + $2h(T) + 1 = 2[1 + \max\{h(T_1), h(T_2)\}] + 1$

$n(T) = 1 + n(T_1) + n(T_2) \underset{IA}{\geq} 1 + [2h(T_1) + 1] + [2h(T_2) + 1] = 2h(T_1) + 2h(T_2) + 2 + 1$

$= 2[1 + h(T_1) + h(T_2)] + 1 \geq 2[1 + \max\{h(T_1), h(T_2)\}] + 1 = 2h(T) + 1$