Typos #6c

Key

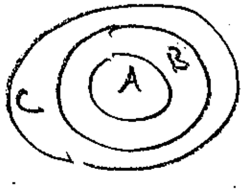
EXAM 2 TCSS 321 /45 Spring 2015

1. [12] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true and briefly justify your answer. Otherwise, circle F and provide a counterexample or brief reasoning. The logic symbols make use of the textbook notation.

(+.5) T F (§2.1 #17) Suppose A, B, and C are sets and that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

start (+.5)
logic (+1)
sense/notation (+1)

venn diagram



(let $a \in A$, we WTS $a \in C$.
or Since $a \in A \subseteq B$ we know $a \in B$.
) Since $a \in B \subseteq C$ we know $a \in C$.

(+.5) T F (Quiz3 #2) The function e^n is $O(200,000n^2)$

$\lim_{n\to\infty} \frac{e^n}{200,000n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{e^n}{400,000n} \overset{L'H}{=} \lim_{n\to\infty} \frac{e^n}{400,000} = \infty$

start (+.5)
def of big O (+1)
computations (+.5)
got it (+.5)

(+.5) T F (matrix wks #1) If M and N are matrices, then $MN = NM$

let M be a 2×3 matrix and N a 3×2 matrix.
Note $M\cdot N$ is a 2×2 matrix and $N\cdot M$ is a 3×3 matrix

start (+.5)
mult def (+1)
computations (+.5)
sense (+.5)

(+.5) T F (HW3 §2.5 #2) The intersection of two countabl~~e~~y infinite sets can never be countabl~~e~~y infinite.

let $A = \mathbb{Z}$ and $B = \mathbb{Z}$. Notice $A \cap B = \mathbb{Z}$

start (+.5)
A + B well def (+.5)
A + B are SS (+.5)
intersection (+.5)
$|A \cap B|$ is SS (+.5)

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Let n be a positive integer. Consider the sets $S_3 = \{a_1, a_2, a_3\}$ and $S_n = \{a_1, a_2, a_3, ...a_n\}$.

(a) [3] (set wks #5) Write down the elements of $\mathcal{P}(S_3)$.

$\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}, \{a_1, a_2, a_3\}$

(+.5)

def of powerset (+1) got all (+1) notation (+.5)

(b) [2] (set wks #6) Find $|\mathcal{P}(S_n)|$.

Notice $|\mathcal{P}(S_3)| = 8$ notation (+.5) know cardinality (+.5)

an element is either included or not, so we have 2^3 choices.

Thus $|\mathcal{P}(S_n)| = 2^n$ got it (+1)

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3. (§2.4 #5) Consider the function f that assigns to each bit string the number of ones in the string minus the number of zeros in the string.

(a) [2] Evaluate $f(01110111)$.

(+1)

$6 - 2 = 4$

(+.5) (+.5)

(b) [2] Identify the range of f.

the integers or $\mathbb{Z}$

(+1)

Knew range def (+.5)
notation (+.5)

start (+.5)
form (+.5)

4. [5] (§3.1 #31) Devise an algorithms that finds the first term of a sequence of integers that equals some previous term in the sequence.

For each entry i in the sequence of integers
Compare the ith integer to those behind it
If there is a match
return i and break out of the for loop
If ith is the last term
return "There is no match"

Idea (+1)

terms defined
within scope

```
Def My Algorithm(list)
for i = 0 to len(list)-2
    for j = i+1 to len(list)-1
        If list[i] = list[j]
            return list[i]
            break
    If i = len(list)-2
        return "No matches"
}
```

precise/detailed
(+1)

return first instance (+.5)
return instance (+.5)

5. [3] (HW5 §2.6 #4) Give a big-O estimate for the number of comparisons between integers that your algorithm in the previous problem uses. Justify your answer.

Given a list of n integers it would make
$(n-1)+(n-2)+(n-3)+\dots+3+2+1$ or
comparisons or
$\frac{(n-1)(n)}{2}$ comparisons.

(+1.5)

b/c there are two nested for loops (+.5) (+.5)
who's index bounds depend on n (+.5)

start (+.5)

So $O(n^2)$

(+1)

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6. Consider the MysterySearch code shown to the right.

```
def MysterySearch(x, list):
    i=0
    while i<=len(list)-1 and x!=list[i]:
        i=i+1
    if i<=len(list)-1:
        location=i
    else:
        location=-1
    return location
```

```
MysterySearch(4,[1,2,3,5,6,7,8,11,12,14,10])
-1
```

```
MysterySearch(11,[1,2,3,5,6,7,8,11,12,14,10])
7
```

(a) [3] (bigOintro wks#4) Give a big-O estimate for the number of comparisons between integers that the algorithms makes. Justify your answer.

+1 O(n)

Knows algorithm +1

Justifies +1

as moves thru the list, checks if x!=list[i] for each i. Worst case is moves thru the entire list. If len(list)=n

(b) [2] (Quiz 3 #6) How does the number of comparisons change in MysterySearch when the inputs change from n to n^2.

answer +1

sense +1

The comparisons would change from about n to n^2

(c) [2] (Quiz 3 #6) Recall that Binary Search is $O(\log n)$. How does the number of comparisons change in ~~MysterySearch~~ Binary Search when the inputs change from n to n^2.

answer +1

sense +1

The comparisons would change from about log n to log n^2 or 2 log n

so double

7. [4] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit. No, doing both questions will not earn you extra credit.

start +.5

understood notation? +.5

sense +1

logic +1

notation +.5

algebra +.5

(a) (§2.4 #11) Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2 \ldots$. Prove that $a_n = 5a_{n-1} - 6a_{n-2}$ for all n greater than or equal to two.

(b) (§1.8 #5) Prove that $\min(x, y) = \frac{x+y-|x-y|}{2}$ whenever x and y are real numbers.

(a) Given $a_n = 2^n + 5 \cdot 3^n$ for $n \in \mathbb{Z}_{\geq 0}$ we want to show $a_n = 5a_{n-1} - 6a_{n-2}$.

Consider $5a_{n-1} - 6a_{n-2}$

$= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2})$

$= 5 \cdot 2^{n-1} + 5^2 \cdot 3^{n-1} - 6 \cdot 2^{n-2} - 6 \cdot 5 \cdot 3^{n-2}$

$= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} + 5^2 \cdot 3^{n-1} - 6 \cdot 5 \cdot 3^{n-2}$

$= 2^{n-2}(5 \cdot 2^1 - 6) + 5 \cdot 3^{n-2}(5 \cdot 3 - 6)$

$= 2^{n-2}(2^2) + 5 \cdot 3^{n-2}(3^2) = 2^n + 5 \cdot 3^n = a_n$ //

(b) We will consider two cases.

1) Assume $\min(x,y) = x$. Then $(x-y) < 0$

So $|x-y| = -(x-y)$.

So $\frac{x+y-|x-y|}{2} = \frac{x+y-(-(x-y))}{2} = \frac{x+\cancel{y}+x-\cancel{y}}{2}$

$= \frac{2x}{2} = x$. Which is what we wanted.

2) If $\min(x,y) = y$, then $|x-y| = x-y$

So $\frac{x+y-|x-y|}{2} = \frac{\cancel{x}+y-\cancel{x}+y}{2} = \frac{2y}{2}$

which is what we wanted.

||

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8. [5] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

(a) (§5.1 #5) Prove that whenever n is a positive integer that,

$$\sum_{i=0}^{n}(2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

(b) (pg 379 #4) Prove that whenever n is a positive integer that,

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\dots\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

Start (1.5)
Sum (1.5)

(a) We will use induction

(.5) Base case: Let $n=1$ then

(1) $\sum_{i=0}^{1}(2i+1)^2 = 1^2+3^2 = 10$ and

$\frac{(1+1)(2\cdot1+1)(2\cdot1+3)}{3} = \frac{2\cdot\cancel{3}\cdot5}{\cancel{3}} = 10$ ✓

Induction: Assume $\sum_{i=0}^{n}(2i+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

(1) (we want to show $\sum_{i=0}^{n+1}(2i+1)^2 = \frac{(n+2)(2(n+1)+1)(2(n+1)+3)}{3}$

So consider the left side

$\sum_{i=0}^{n+1}(2i+1)^2 = \underbrace{1^2+3^2+5^2+\dots+(2n+1)^2}+(2(n+1)+1)^2$ by the assump.

$= \frac{(n+1)(2n+1)(2n+3)}{3} + (2n+3)^2$

algebra (1)
sense (1.5)

$= (2n+3)\left[\frac{(n+1)(2n+1)}{3} + \frac{2n+3}{1}\right]$

$= (2n+3)\left[\frac{2n^2+3n+1+3(2n+3)}{3}\right]$

$= (2n+3)\frac{(2n^2+9n+10)}{3}$

$= \frac{(2n+3)(n+2)(2n+5)}{3}$

$= \frac{(n+2)(2(n+1)+1)(2(n+1)+3)}{3}$ which is what we wanted

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Start (1.5)
Sum (1.5)

(b) We will use induction

(.5) Base case: Let $n=1$ then $\frac{1}{2(1)+1} = \frac{1}{3}$

(1) and $\frac{1}{(2(1)+1)(2\cdot1-1)} = \frac{1}{3\cdot1} = 1/3$ ✓

Induction: Assume $\frac{1}{1\cdot3}+\dots+\frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

we WTS (1) $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\dots+\frac{1}{(2(n+1)-1)(2(n+1)+1)} = \frac{n+1}{2(n+1)+1}$

or rather that $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\dots+\frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}$.

So consider the left side

$\underbrace{\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\dots+\frac{1}{(2n-1)(2n+1)}} + \frac{1}{(2n+1)(2n+3)}$

$= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)}$ by our assumption

algebra (1)
sense (1.5)

$= \frac{n(2n+3)}{(2n+1)(2n+3)} + \frac{1}{(2n+1)(2n+3)}$ getting a common den

$= \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{\cancel{(2n+1)}(n+1)}{\cancel{(2n+1)}(2n+3)}$

$= \frac{n+1}{2n+3} = \frac{n+1}{2(n+1)+1}$

which is what we wanted to show.