TMATH 307 MATRIX ALGEBRA (PROFESSOR VANDERPOOL)

SUBSTITUTE OF THE DAY: PROFESSOR JENNIFER QUINN

PLAN

TUESDAY	11/25	CLASS §4.4
THURSDAY	11/27	NO CLASS 
TUESDAY	12/2	CLASS PRESENTATIONS HW DUE: §4.1, 4.2, 4.3

SUGGESTED PROBLEMS: § 4.4: 5, 7, 9, 11, 13, 15, 41, 43

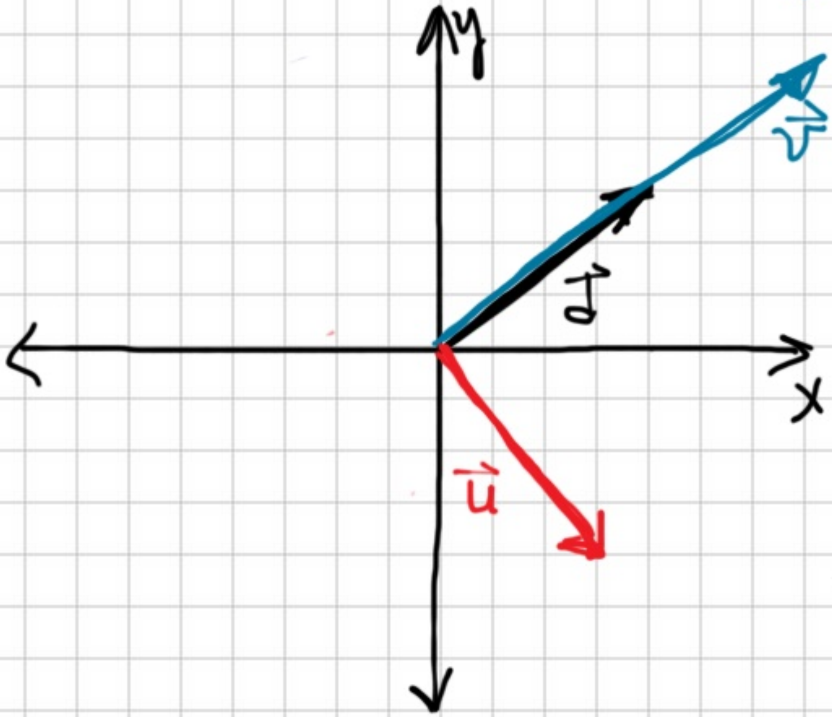
¿HOMEWORK QUESTIONS? UNTIL ~ 1:10 pm

HOMEWORK QUESTIONS

$A = \begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix}$

FIND projection onto line through origin w/ DIRECTION VECTOR $\vec{d} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

FIND E-VALUES & E-VECTORS of A GEOMETRICALLY!



- vector in same direction as $\vec{d}$, say $\vec{v}$ $(\lambda=1)$

- vector $\perp$ to $\vec{d}$.

$\vec{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$ $(\lambda=0)$

TEST $A\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ ON YOUR OWN

§4.4 DIAGONALIZATION + SIMILARITY

RECALL EIGENVALUES. (RIGHT?)

FIND EIGENVALUES OF $\begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix}$

$$\det\begin{bmatrix}1-\lambda & 3\\ 2 & 2-\lambda\end{bmatrix} = (1-\lambda)(2-\lambda)-6$$

$$=2-3\lambda+\lambda^2-6$$

$$=\lambda^2-3\lambda-4$$

$$=(\lambda-4)(\lambda+1)$$

SO EIGENVALUES ARE

$$\lambda=4,-1$$

DEFINITION LET A & B BE $n\times n$ MATRICES. WE SAY A IS SIMILAR TO B IF THERE EXISTS AN INVERTIBLE $n\times n$ MATRIX P SUCH THAT $P^{-1}AP=B$. WE WRITE $A\sim B$.

DEFINITION LET A & B BE $n \times n$ MATRICES. WE SAY A IS SIMILAR TO B IF THERE EXISTS AN INVERTIBLE $n \times n$ MATRIX P SUCH THAT $P^{-1}AP = B$.
WE WRITE $A \sim B$.

NOTES/THEOREM (SIMILARITY FORMS A RELATION ON $n \times n$ MATRICES) SPECIFICALLY IF A, B, C ARE $n \times n$ MATRICES

① $A \sim A$ (REFLEXIVE) b/c $I^{-1}AI = A$

② IF $A \sim B$, THEN $B \sim A$ (SYMMETRIC) b/c $\underset{A \sim B}{P^{-1}AP = B} \Rightarrow \underset{B \sim A}{(P^{-1})^{-1}B\overset{\text{DIFFERENT "P"}}{P^{-1}} = A}$

③ IF $\underbrace{A \sim B}_{\text{HYPOTHESIS}}$ AND $\underbrace{B \sim C}_{\text{HYPOTHESIS}}$, THEN $A \sim C$. (TRANSITIVITY) b/c IF $P^{-1}AP = B$ AND $Q^{-1}BQ = C$
THEN $C = \underbrace{Q^{-1}P^{-1}}_{R^{-1}}A\underbrace{PQ}_{R} = (PQ)^{-1}A(PQ)$
$A \sim C$.

SO? WHAT DOES THIS HAVE TO DO WITH CHAPTER 4?

THEOREM LET A & B BE $n \times n$ MATRICES WITH $A \sim B$. THEN

(a) $\det A = \det B$
(b) A IS INVERTIBLE $\Leftrightarrow$ B IS INVERTIBLE
(c) A AND B HAVE THE SAME RANK
d) A AND B HAVE THE SAME CHARACTERISTIC POLYNOMIAL
e) A AND B HAVE SAME EIGENVALUES

EXAMPLE CONSIDER $A=\begin{bmatrix}1&3\\2&2\end{bmatrix}$, $P=\begin{bmatrix}1&3\\1&-2\end{bmatrix}$, AND $D=\begin{bmatrix}4&0\\0&-1\end{bmatrix}$

VERIFY $A\sim D$ BY COMPUTING $P^{-1}AP$

$$\underbrace{\frac{1}{-2-3}\begin{bmatrix}-2&-3\\-1&1\end{bmatrix}}_{P^{-1}}\underbrace{\begin{bmatrix}1&3\\2&2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1&3\\1&-2\end{bmatrix}}_{P}=\frac{-1}{5}\begin{bmatrix}-2&-3\\-1&1\end{bmatrix}\begin{bmatrix}4&-3\\4&2\end{bmatrix}=-\frac{1}{5}\begin{bmatrix}-20&0\\0&5\end{bmatrix}=\begin{bmatrix}4&0\\0&-1\end{bmatrix}=D$$

EIGENVECTOR FOR $\lambda=4$

EIGENVECTOR FOR $\lambda=-1$

$$\underbrace{\begin{bmatrix}1&3\\2&2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\boxed{\begin{matrix}1\\1\end{matrix}}&\boxed{\begin{matrix}3\\-2\end{matrix}}\end{bmatrix}}_{P}=\begin{bmatrix}\boxed{\begin{matrix}4\\4\end{matrix}}&\boxed{\begin{matrix}-3\\2\end{matrix}}\end{bmatrix}$$

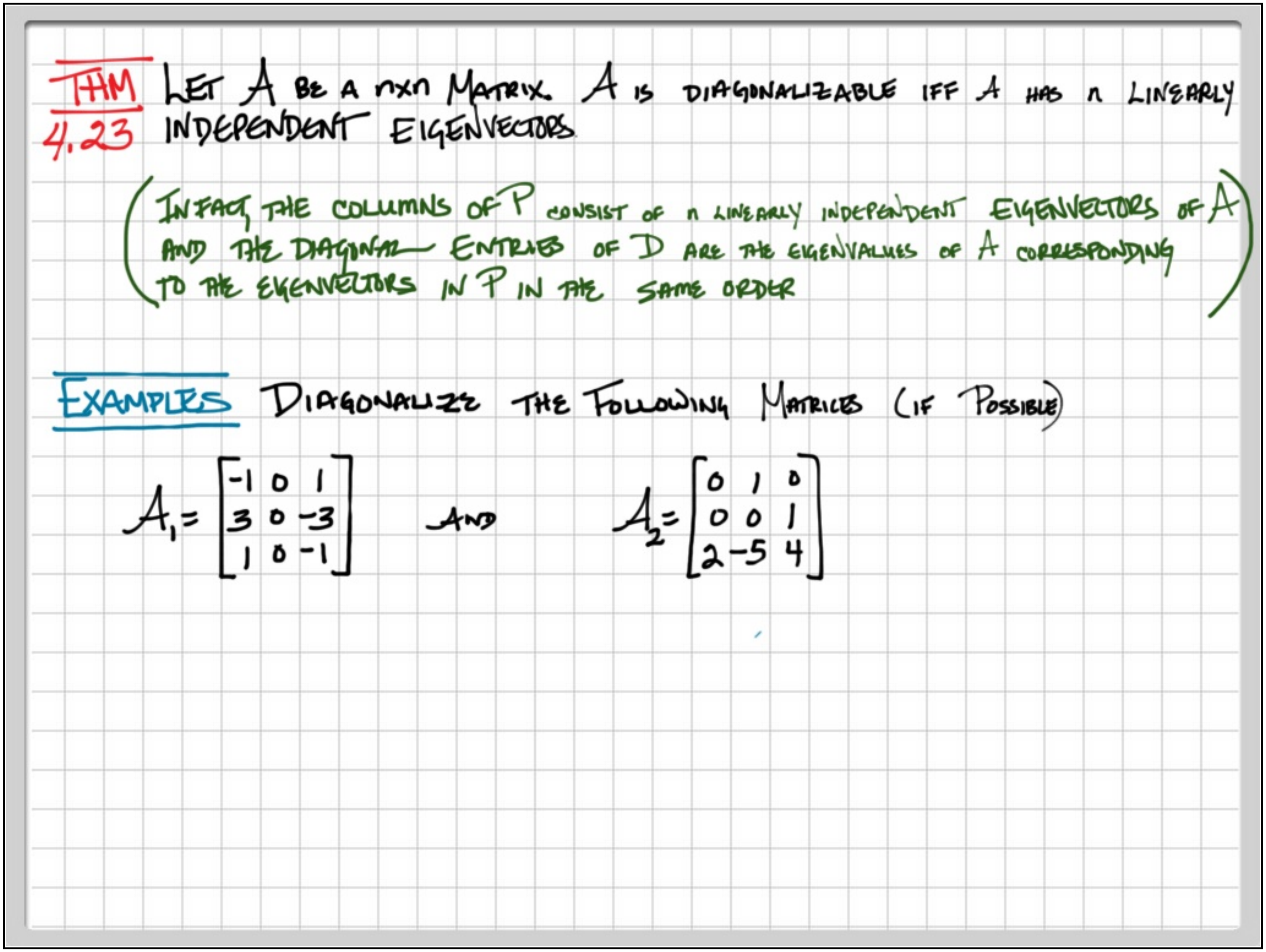
WOULD $P=\begin{bmatrix}1&2\\1&2\end{bmatrix}$ WORK?

NO. NOT INVERTIBLE.
COLUMNS NOT LIN. IND.

DEFINITION AN $n\times n$ MATRIX A IS **DIAGONALIZABLE** IF THERE IS A DIAGONAL MATRIX D SUCH THAT $A\sim D$.

¿DO YOU NOTICE ANYTHING SPECIAL ABOUT THE DIAGONAL VALUES IN D FROM THE EXAMPLE TODAY?

EIGENVALUES.
IN FACT, COLUMNS OF P ARE EIGENVECTORS.

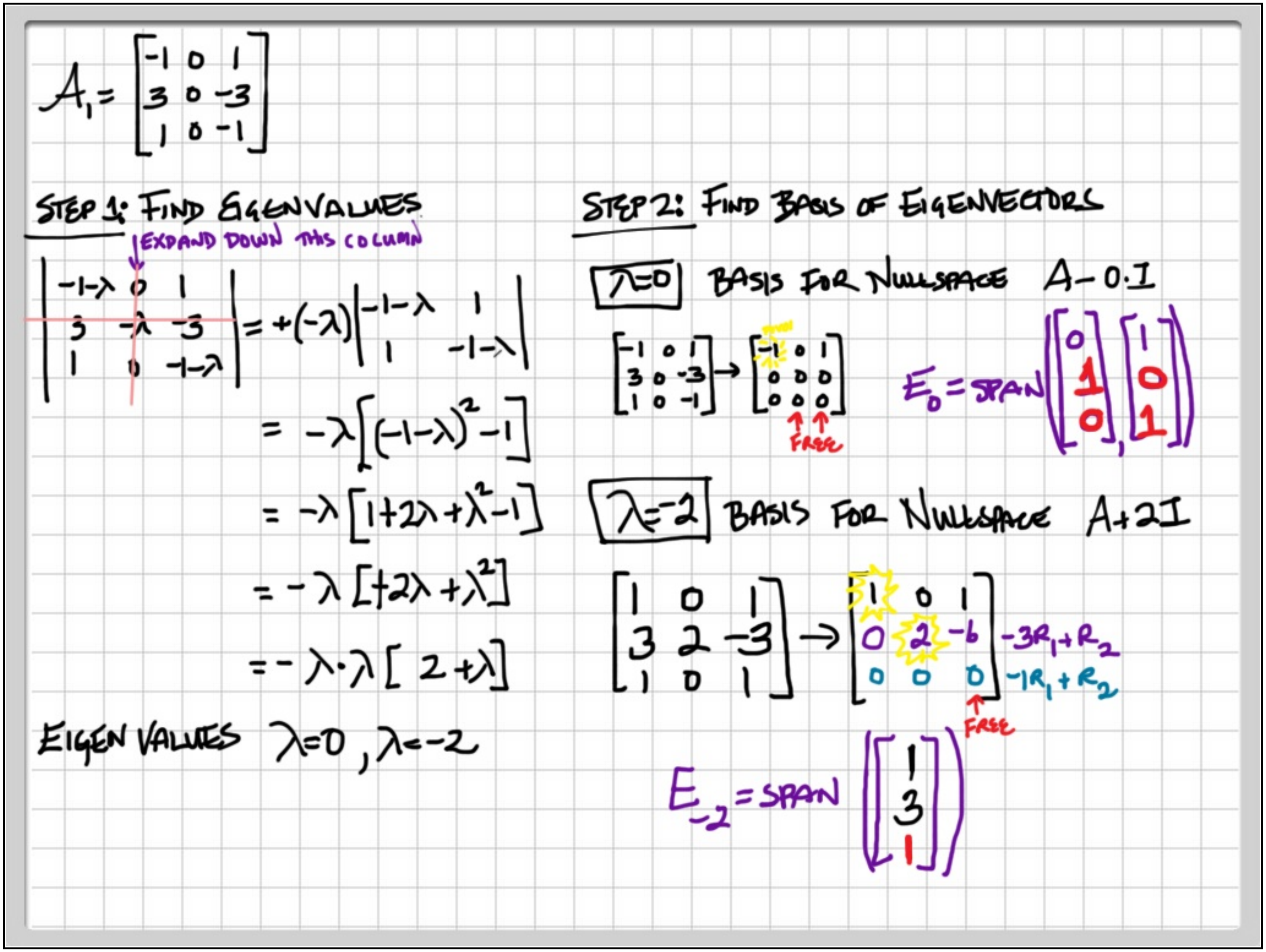


THM 4.23 LET A BE A $n \times n$ MATRIX. A IS DIAGONALIZABLE IFF A HAS n LINEARLY INDEPENDENT EIGENVECTORS

(IN FACT THE COLUMNS OF P CONSIST OF n LINEARLY INDEPENDENT EIGENVECTORS OF A AND THE DIAGONAL ENTRIES OF D ARE THE EIGENVALUES OF A CORRESPONDING TO THE EIGENVECTORS IN P IN THE SAME ORDER)

EXAMPLES DIAGONALIZE THE FOLLOWING MATRICES (IF POSSIBLE)

$$A_1 = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{AND} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$



$$A_1 = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

STEP 1: FIND EIGENVALUES

EXPAND DOWN THIS COLUMN

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = +(-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= -\lambda\left[(-1-\lambda)^2-1\right]$$

$$= -\lambda\left[1+2\lambda+\lambda^2-1\right]$$

$$= -\lambda\left[+2\lambda+\lambda^2\right]$$

$$= -\lambda\cdot\lambda\left[2+\lambda\right]$$

EIGEN VALUES $\lambda=0$, $\lambda=-2$

STEP 2: FIND BASIS OF EIGENVECTORS

$\boxed{\lambda=0}$ BASIS FOR NULLSPACE $A-0\cdot I$

$$\begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \quad\uparrow\ \uparrow \\ \text{FREE} \end{array}$$

$$E_0 = \text{SPAN}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

$\boxed{\lambda=-2}$ BASIS FOR NULLSPACE $A+2I$

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ -3R_1+R_2 \\ -1R_1+R_2 \end{array} \\ \qquad\quad\uparrow \\ \qquad\quad\text{FREE} \end{array}$$

$$E_{-2} = \text{SPAN}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right)$$

$$A_1 = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

STEP 3: CREATE P, D

$$P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

STEP 4: CHECK $P^{-1}AP = D$

<EASIER TO CHECK $AP = PD$>

$$A_2 = \begin{bmatrix}0&1&0\\0&0&1\\2&-5&4\end{bmatrix}$$

STEP 1: FIND EIGENVALUES

$$\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\2&-5&4-\lambda\end{vmatrix} \leftarrow = -\lambda\begin{vmatrix}-\lambda&1\\-5&4-\lambda\end{vmatrix} - (1)\begin{vmatrix}0&1\\2&4-\lambda\end{vmatrix}$$

$$= -\lambda[(-4\lambda+\lambda^2+5)] - [0-2]$$

$$= -\lambda^3+4\lambda^2-5\lambda+2$$

$\lambda=1$ ROOT

$$= (\lambda-1)(-\lambda^2+3\lambda-2)$$

$$= (\lambda-1)(-\lambda+1)(\lambda-2)$$

$$= -(\lambda-1)^2(\lambda-2)$$

$$\begin{array}{c|cccc} & -1 & 4 & -5 & 2 \\ 1 & & -1 & 3 & -2 \\ \hline & -1 & 3 & -2 & 0 \end{array}$$

EIGENVALUES

$\lambda = 1, 1, 2$

STEP 2: FIND BASIS OF EIGENVECTORS

$$\boxed{\lambda=1}$$

$$\begin{bmatrix}-1&1&0\\0&-1&1\\2&-5&3\end{bmatrix} \rightarrow \begin{bmatrix}1&-1&0\\0&-1&1\\0&-3&3\end{bmatrix} \xrightarrow{} {\scriptstyle 2R_1+R_3}$$

$$\rightarrow \begin{bmatrix}1&-1&0\\0&-1&1\\0&0&0\end{bmatrix}$$

$$\qquad\qquad\qquad\uparrow$$

$$E_1 = \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle$$

$\boxed{\lambda=2}$

$$\begin{bmatrix}-2&1&0\\0&-2&1\\2&-5&2\end{bmatrix}\rightarrow\begin{bmatrix}-2&1&0\\0&-2&1\\0&-4&2\end{bmatrix}\rightarrow\begin{bmatrix}-2&1&0\\0&-2&1\\0&0&0\end{bmatrix}$$

$$\begin{array}{r}\uparrow\\ \text{FREE}\end{array}$$

$$E_2=\left\langle\begin{bmatrix}1\\2\\4\end{bmatrix}\right\rangle$$

ONLY 2 EIGENVECTORS, NOT 3.

CANNOT BE DIAGONALIZED

THM 4.23 LET A BE A $n \times n$ MATRIX. A IS DIAGONALIZABLE IFF A HAS n LINEARLY INDEPENDENT EIGENVECTORS.

PROOF (⇒) SUPPOSE A IS DIAGONALIZABLE. THEN A IS SIMILAR TO A DIAGONAL MATRIX D VIA $P^{-1}AP=D$. THIS IS EQUIVALENT TO $AP=PD$. LET THE COLUMNS OF P be $\overline{p_1}, \overline{p_2}, \dots, \overline{p_n}$ AND THE DIAGONAL ENTRIES OF D BE $\lambda_1, \lambda_2, \dots, \lambda_n$.

$$P = \left[\overline{p_1} \middle| \overline{p_2} \middle| \cdots \middle| \overline{p_n}\right] \quad \text{AND} \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

BY ASSUMPTION $AP=PD$

$$A\left[\overline{p_1} \middle| \overline{p_2} \middle| \cdots \middle| \overline{p_n}\right] = \left[\overline{p_1} \middle| \overline{p_2} \middle| \cdots \middle| \overline{p_n}\right] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

$$\left[A\overline{p_1} \middle| \boxed{A\overline{p_2}} \middle| \cdots \middle| A\overline{p_n}\right] = \left[\lambda_1\overline{p_1} \middle| \boxed{\lambda_2\overline{p_2}} \middle| \cdots \middle| \lambda_n\overline{p_n}\right]$$

EQUATING COLUMNS IN GENERAL WE GET $A\overline{p_i} = \lambda_i \overline{p_i}$ FOR $i=1,2,\dots,n$. SO $\overline{p_i}$ IS an EIGENVECTOR CORRESPONDING TO λ_i.

SINCE P IS INVERTIBLE, ITS COLUMNS ARE LINEARLY INDEPENDENT.

($\Leftarrow$) SUPPOSE A HAS n LINEARLY INDEPENDENT EIGENVECTORS $\overline{p_1}, \overline{p_2}, \dots, \overline{p_n}$ AND ASSOCIATED EIGENVALUES $\lambda_1, \lambda_2, \dots, \lambda_n$.

THEN $A\overline{p_i} = \lambda_i \overline{p_i}$ for $i = 1, 2, \dots, n$.

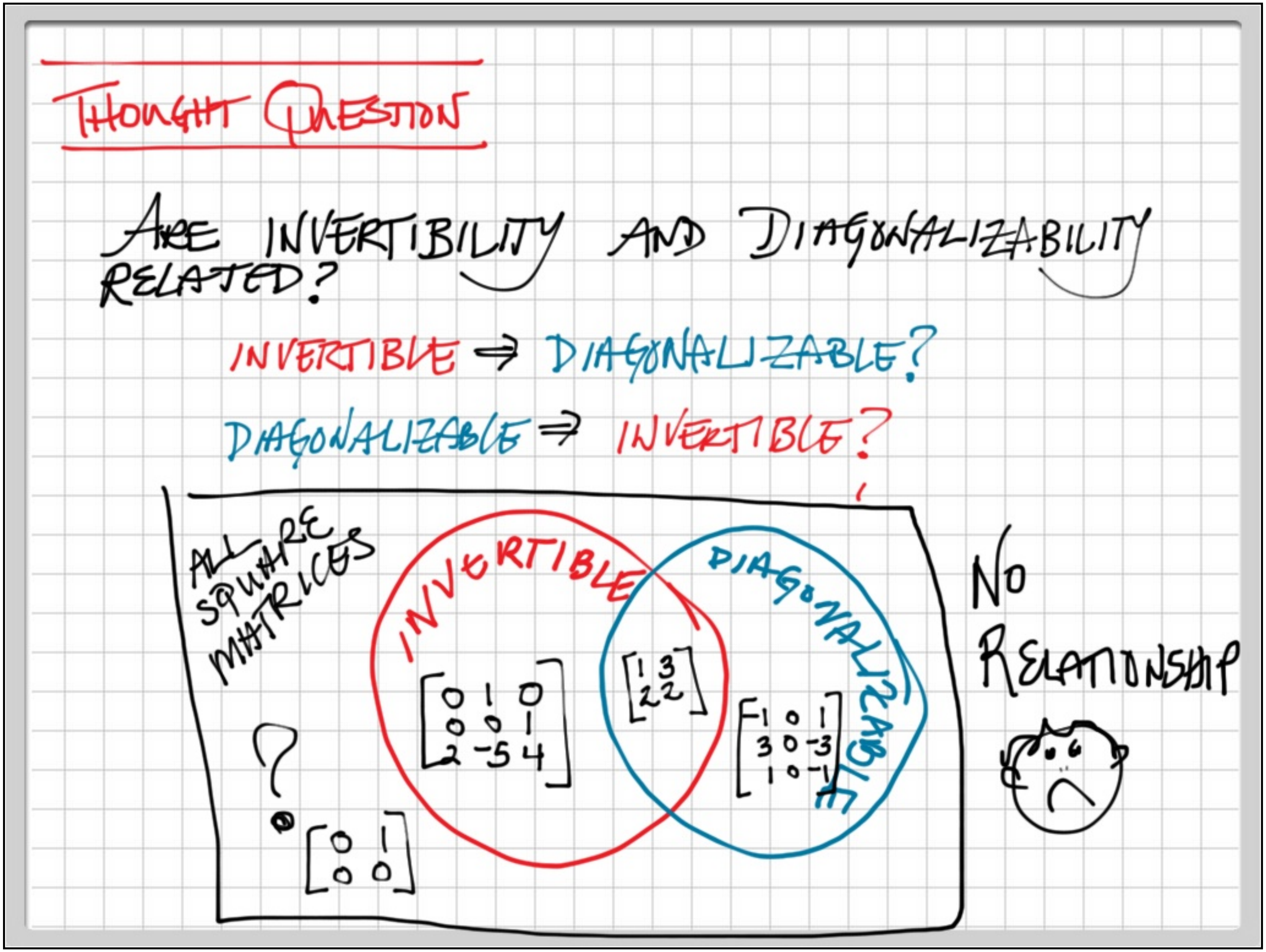
DEFINE $P = [\overline{p_1} | \overline{p_2} | \cdots | \overline{p_n}]$.

THEN $$AP = [\lambda_1 \overline{p} | \lambda_2 \overline{p} | \cdots | \lambda_n \overline{p}] = P \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

SO $AP = PD$ WHERE D IS DIAGONAL

SINCE THE COLUMN OF P ARE LINEARLY INDEPENDENT, WE CONCLUDE $P^{-1}AP = D$. So $A \sim D$ AND A IS DIAGONALIZABLE

$\square$.



THOUGHT QUESTION

ARE INVERTIBILITY AND DIAGONALIZABILITY RELATED?

INVERTIBLE ⇒ DIAGONALIZABLE?

DIAGONALIZABLE ⇒ INVERTIBLE?