EXAM 2 # TMath 308 Autumn 2014

NAME: Key

True/False: If the statement is *always* true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. [4] Given any matrices A, B, then $A + B$ is well defined.

(+1) False

If A and B are not the same sizes, A+B is not well defined

Start (+.5)
logic (+1)
sense/style (+.5)

ex $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

matrix addition (+1)

2. [4] Given any matrices A, B, then $A * B = B * A$.

(+1) False

ex Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Start (+.5)
logic (+1)
sense/style (+.5)

matrix mult (+1)

Notice $\underset{3\times1}{A} \cdot \underset{1\times2}{B} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{bmatrix}$

and $\underset{1\times2}{B} \cdot \underset{3\times1}{A}$ isn't even defined

3. [4] The main diagonal of a skew-symmetric matrix must consist of entirely of zeros.

(+1) True

Let A be skew-symmetric. skew-sym (+.5)
Then $-A = A^T$. use def (+.5)
We can consider the entries on the diagonal. Then $-A = A^T$.
$\Rightarrow -a_{ii} = a_{ii}$ for all the entries on the diagonal. The only # with this p

Start (+.5)
logic (+1)
sense style (+.5)

4. [4] Given any matrices A and B such that $A * B$ equals the zero matrix, then either A or B must be the zero matrix.

(+1) False

Consider $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

then $A \cdot B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

but neither A nor B were the zero matrix

start (+.5)
logic (+1)
sense/style (+1.5)

null (+.5)
noninvertible (+.5)

5. [4] Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. The vector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is in null(A).

(+1) True

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

def of null (+1)

(recall $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is in null(A) if $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$.)

start (+.5)
logic (+1)
sense/style (+.5)

6. [4] The columns in a 3×3 matrix A, form a basis for col(A).

(+1) False

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

def of basis (+1)
def of col(A) (+.5)

start (+.5)
logic (+.5)
style (+.5)

recall $\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

$= \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$

a linearly indep. set that spans is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

7. Let a, b, and c be real, nonzero numbers and let A, B, and C be matrices defined by:

$$A = \begin{bmatrix} a & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ b & 0 \\ 3 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} c & 0 \\ 2 & 1 \end{bmatrix}.$$

Compute the following if possible.

(a) [2] $A^{\mathrm{T}} + B$

$$\begin{bmatrix} a & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ b & 0 \\ 3 & 0 \end{bmatrix}$$

+1

+1 $\begin{bmatrix} a & 0 \\ b & 0 \\ 3 & 1 \end{bmatrix}$

(b) [4] CA $\quad (2\times 2)\times(2\times 3)$

$$\begin{bmatrix} c & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

dimensions +1

$$\begin{bmatrix} ac^{+.5} & 0^{+.5} & 0^{+.5} \\ 2a-1^{+.5} & 0^{+.5} & 1^{+.5} \end{bmatrix}$$

(c) [3] C^3B

$$\underbrace{(2\times 2)\times(2\times 2)(2\times 2)}\times(3\times 2)$$

$(2\times 2)\times(3\times 2)$

dimensions +1

powers +1

not compatible sizes

+1 { multiplication is not defined here?

(d) [3] C^{-1}

$$\frac{1}{\det(C)}\begin{bmatrix} 1 & -0 \\ -2 & c \end{bmatrix} = \underbrace{\frac{1}{c-0}}_{+1}\begin{bmatrix} 1 & 0 \\ -2 & c \end{bmatrix}^{+2}$$

$$= \frac{1}{c}\begin{bmatrix} 1 & 0 \\ -2 & c \end{bmatrix} = \begin{bmatrix} 1/c & 0 \\ -2/c & 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} c & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1/c & 0 \\ -2/c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/c & 0 \\ -2/c & 1 \end{bmatrix}\begin{bmatrix} c & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

or

+1.5 $\left\{ \left[\begin{array}{cc|cc} c & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{c}R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1/c & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \right.$

$$\xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1/c & 0 \\ 0 & 1 & -2/c & 1 \end{array}\right]$$

Row reduce +1.5

So $\begin{bmatrix} 1/c & 0 \\ -2/c & 1 \end{bmatrix}$

8. [4] Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find the conditions on a, b,c and d such that M commutes multiplicatively with the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

We want to find a, b, c, d so that

(+1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

mult (.5) $\Rightarrow \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$

set up system (+1)

$$\Rightarrow \begin{cases} a+3b = a+2c \\ c+3d = 3a+4c \\ 2a+4b = b+2d \\ 2c+4d = 3b+4d \end{cases} \Rightarrow \begin{cases} 3b-2c = 0 \\ -3a - 3c+3d = 0 \\ 2a+3b - 2d = 0 \\ -3b+2c = 0 \end{cases}$$

$\rightarrow$ we need to solve a sys of equations?

$$\left[\begin{array}{cccc|c} 0 & 3 & -2 & 0 & 0 \\ -3 & 0 & -3 & 3 & 0 \\ 2 & 3 & 0 & -2 & 0 \\ 0 & -3 & 2 & 0 & 0 \end{array}\right]$$

(+1) wishes she had a calculator

$-\frac{1}{3}R_2$, $R_1 \leftrightarrow R_2$, $R_1+R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} a = t-r \\ b = 2/3\, r \\ c = r \\ d = t \end{array}$$

or $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} r \;\middle|\; t, r \in \mathbb{R} \right\}$

sol set (+.5) sol set correct

9. [4] Consider the set $S = \{[x, y, z] \in \mathbb{R}^3 | z = 2x \text{ and } y = 0\}$.
Prove S is a subspace of $\mathbb{R}^3$ or give a counterexample to show that it does not.

understand set/S (+1)

$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = 2x \text{ and } y = 0 \right\} = \left\{ \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} \middle| x \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} x \middle| x \in \mathbb{R} \right\}$

is a "line" with direction $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ thru the origin.

We verify the three properties for a set to be a subspace

1) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot 0$ so $\vec{0} \in S$ ✓

(+.5) know condition
(+.5) check

2) Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} \in S$. Then $\exists\, s, t \in \mathbb{R}$ such that

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} s$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} t$. Notice then that

(+.5) know condition
(+.5) check

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} t = (s+t)\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in S$. ✓

3) Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in S$, Then $\exists\, s \in \mathbb{R} \ni \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} s$.

(+.5) know c
(+.5) check

Let $c \in \mathbb{R}$ Notice $c\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} s = cs\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in S$ ✓

10. Provide a lecture on induction. That is:

(a) [3] explain the logic used in an induction proof.

(b) [1] identify the essential elements in an induction proof. and

(c) [2] provide an example induction proof with the above essential elements.

(a) (1) Induction relies on there being a clear process from one discrete step to another. If you need to prove a property & you can show

1) at least one step exhibits the property

2) your property is maintained from one discrete step to the next

start +.5
style +.5
logic +1

This means your property will always be? Yay?

Note: we can prove induction using the well ordering Principal ...

(b) +.5 Base case: where you show your property holds

+.5 Induction: Assume your property holds for n, Show your property holds for $n+1$

(c) There are so many good examples of induction. Here's a few:

+.5 (i) We will prove $1+2+\dots+n = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

+.5 Base Case: Notice $1 = \frac{1(1+1)}{2}$ ✓

+1 Induction: Assume $1+2+\dots+k = \frac{k(k+1)}{2}$.

We would like to show that

$1+2+\dots+k+(k+1) = \frac{(k+1)(k+1+1)}{2}$

Consider $1+2+\dots+k+(k+1)$

$= \frac{k(k+1)}{2} + k+1$ (by our assumption).

$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$ ✓

(ii) We will prove $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for all integers $n \geq 1$

Base Case: Notice $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ✓

Induction: Assume that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Consider

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$ ✓