2
24
18
8
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Typos: #10
#1

EXAM 1 TMath 308 Fall 2014

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NAME: Key

True/False: If the statement is *always* true, give a *brief* explanation of why it is (not a "formal" proof!). If the statement is false, give a counterexample.

1. [4] If $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ are in $\mathbb{R}^n$, where $n \geq 2$, then $\overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{w}$ makes sense

(+1) False

dot prod (+.5)
add vector (+.5)

start (+.5)
logic (+1)
sense/style (+.5)

The statement makes no sense. $\vec{u} \cdot \vec{v}$ is a scalar which cannot be added to the vector $\vec{w}$.

2. [4] The plane defined by $x + 2y + 3z = 1$ is parallel to the line defined by
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ where $t \in \mathbb{R}$.

(+.5) plane/normal vector
(+.5) line/directional vector

(+1) False

start (+.5)
logic (+1)
sense/style (+.5)

The plane $x+2y+3z=1$ is equivalent to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$ so the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{\eta}$ is normal to the plane.

Note the directional vector in the line is the same as $\vec{\eta}$

$\therefore$ the line is $\perp$ to the plane.

3. [4] If $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ are in $\mathbb{R}^n$, where $n \geq 2$, then $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \underline{u} \cdot \overrightarrow{w}$.

general (+.5)
dot/add (+.5)

(+1) True

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$

start (+.5)
logic (+1)
sense/style (+.5)

Then $\vec{u} \cdot (\vec{v} + \vec{w}) = u_1(v_1+w_1) + u_2(v_2+w_2) + \dots + u_n(v_n+w_n)$

$= u_1v_1 + u_2v_2 + \dots + u_nv_n + u_1w_1 + u_2w_2 + \dots + u_nw_n$

$1 = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

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4. [4] If a system of linear equations has more variables than equations, then the system has infinitely many solutions.

(+1) False   Consider $\left\{\begin{array}{l} x+y+z+w=1 \\ 2x+2y+2z+2w=5 \end{array}\right\}$ (+.5) # of var vs # of eq (+.5) system sol

start (+.5)
logic (+1)
sense/style (+.5)

note # variables $= 4 >$ # of equations

but there are no solutions.

5. [4] The vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ are in span($\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{u}+\overrightarrow{w}$).

(+1) True

$\vec{u} \in \text{span}(\vec{u}, \vec{v}, \vec{u}+\vec{w})$

b/c $\vec{u} = 1\vec{u} + 0\vec{v} + 0(\vec{u}+\vec{w})$

$\vec{v} \in \text{span}(\vec{u}, \vec{v}, \vec{u}+\vec{w})$

b/c $\vec{v} = 0\vec{u} + 1\vec{v} + 0(\vec{u}+\vec{w})$

start (+.5)
logic (+.5)
sense/style (+.5)

$\vec{w} \in \text{span}(\vec{u}, \vec{v}, \vec{u}+\vec{w})$

b/c $\vec{w} = (-1)\vec{u} + 0\vec{v} + 1(\vec{u}+\vec{w})$

def of lin. combo (+.5)
def of span (+1)

6. [4] The vector $\begin{bmatrix}1\\2\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\-1\end{bmatrix}$

(+1) True

Is there $a, b \in \mathbb{R}$ so that   def of lin combo (+1)

$\begin{bmatrix}1\\2\end{bmatrix} = a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}2\\-1\end{bmatrix}$ set up system/process (+.5)

start (+.5)
logic (+.5)
sense/style (+.5)

ie $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \end{array}\right]$

so $-5\begin{bmatrix}1\\-1\end{bmatrix} + 3\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ ✓

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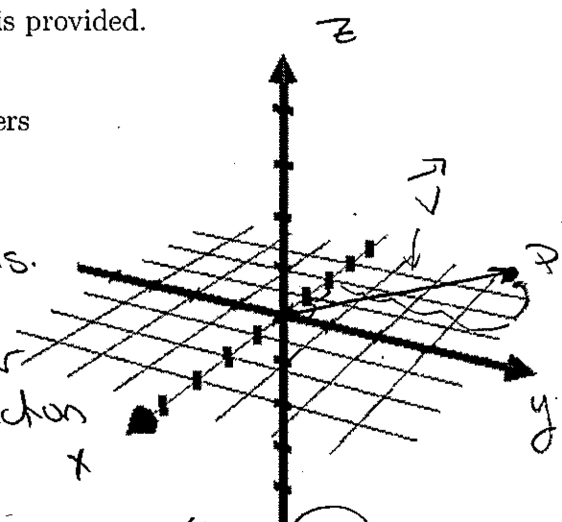
Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

7. (a) [1] Identify the point $P = (-2, 4, 1)$ on the axis provided.

   (b) [1] Identify the vector $\overrightarrow{v} = \langle -2, 4, 1 \rangle$.

   (c) [2] Describe the difference between your answers for part (a) and (b).



The point P is fixed with respect to the coordinent axis.

The vector $\vec{v}$ can however move as long as the direction and magnitude remain unchanged.

true (+.5) difference (+1) sense/style (+.5)

   (d) [2] Compute $\overrightarrow{v} + 3\,\overrightarrow{i}$.

$$\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

translate to #'s (+.5) (+.5) (+1)

   (e) [2] Write down the equation of a line parallel to $\overrightarrow{v}$ that passes through P.

(+.5) (+.5) (+.5) line (+.5)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} t \quad \text{where } t \in \mathbb{R}$$

or

$x = -2 + -2t \qquad y = 4 + 4t \qquad z = 1 + t$

or

$\frac{x+2}{-2} = t = \frac{y-4}{4} = z - 1$ so $\frac{x+2}{-2} = \frac{y-4}{4} = z - 1$

or

$-2x - 4 = y - 4 = 4z - 4$

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8. [2] Create a system of linear equations that has only one solution.

start (+.5)
only 1 solution (+.5)
system (+1)

$$\begin{Bmatrix} x = 5 \\ y = 0 \end{Bmatrix}$$

9. [6] Let a and b be nonzero real numbers and consider the augmented matrix: $\begin{bmatrix} 3 & 6 & 0 & -3 & : 18 \\ 0 & 0 & b & -6 & : 3 \\ a & 0 & -b & 0 & : 6a \end{bmatrix}$

Use Gaussian Elimination, or Gauss-Jordan Elimination, or any series of elementary row operations that make sense to solve the system.

start (+.5)
ERO (+1.5)
algorithm that works (+2)

$$\left[\begin{array}{cccc|c} 3 & 6 & 0 & -3 & 18 \\ 0 & 0 & b & -6 & 3 \\ a & 0 & -b & 0 & 6a \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 6 \\ 0 & 0 & b & -6 & 3 \\ a & 0 & -b & 0 & 6a \end{array}\right] \xrightarrow{R_3 - aR_1 \to R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 6 \\ 0 & 0 & b & -6 & 3 \\ 0 & -2a & -b & a & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 6 \\ 0 & -2a & -b & a & 0 \\ 0 & 0 & b & -6 & 3 \end{array}\right] \xrightarrow{R_3 + R_2 \to R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 6 \\ 0 & -2a & 0 & a-6 & 3 \\ 0 & 0 & b & -6 & 3 \end{array}\right] \xrightarrow[\frac{1}{b}R_3]{\frac{-1}{2a}R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 6 \\ 0 & 1 & 0 & \frac{a-6}{-2a} & \frac{-3}{2a} \\ 0 & 0 & 1 & -6/b & 3/b \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 + \frac{a-6}{a} & 6 + 3/a \\ 0 & 1 & 0 & \frac{a-6}{-2a} & -3/2a \\ 0 & 0 & 1 & -6/b & 3/b \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -6/a & \frac{6a+3}{a} \\ 0 & 1 & 0 & \frac{6-a}{2a} & \frac{-3}{2a} \\ 0 & 0 & 1 & -6/b & \frac{3}{b} \end{array}\right]$$

$x - 6/a\, w = \frac{6a+3}{a}$

$y + \frac{6-a}{2a} w = -\frac{3}{2a}$

$z - \frac{6}{b} w = 3/b$

let $w = r$ where $r \in \mathbb{R}$ then $x = \frac{6a+3}{a} + \frac{6}{a} r$ $\quad y = \frac{-3}{2a} - \frac{6-a}{2a} r$ and $z = \frac{3}{b} + \frac{6}{b} r$

yuk

translate answer (+2)

10. [2] Let a and b be nonzero real numbers and solve the following system of linear equations:

this has the same solution set as #9.

start (+.5)
connect to #9 (+1)
sense/style (+.5)

$$3x + 6y - 3w = 18$$
$$bz - 6w = 3$$
$$ax - bz = 6a$$

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11. [3] Identify a topic that did not appear on this exam and then construct a question about/for this topic. (Yes, I'd like you to help me write your final exam!)

start (+.5)
not on exam (+1)
well formed (+1)
sense/style (+.5)

find the rank of a matrix
how many free variables there are in a lin. sys.
the angle between two given vectors

T/F homogeneous matrices have no solutions
find the distance between a point & a line.
is a given set of vectors linearly dependent?
show two matrices are row equivalent.
find the line of intersection between 2 planes.

12. [5] *Prove* $\text{proj}_{\overrightarrow{u}}(\text{proj}_{\overrightarrow{u}}(\overrightarrow{v})) = \text{proj}_{\overrightarrow{u}}(\overrightarrow{v})$ where $\overrightarrow{u}$ and $\overrightarrow{v}$ are in $\mathbb{R}^n$.

(algebra method)

(+.5) Recall that we can compute $\text{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$.
where $\left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)$ is a scalar.

Then $\text{proj}_{\vec{u}}(\text{proj}_{\vec{u}}(\vec{v})) = \text{proj}_{\vec{u}}\left(\left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}\right)$

$$= \frac{\vec{u}\cdot\left(\left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}\right)}{\vec{u}\cdot\vec{u}}\vec{u} \quad (+1)$$

$$= \frac{\left(\frac{\vec{u}\cdot\vec{v}}{\cancel{\vec{u}\cdot\vec{u}}}\right)\cancel{\vec{u}\cdot\vec{u}}}{\vec{u}\cdot\vec{u}}\vec{u} \quad \text{properties } (+1)$$

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \text{proj}_{\vec{u}}(\vec{v}).$$

Thus $\text{proj}_{\vec{u}}(\text{proj}_{\vec{u}}(\vec{v})) = \text{proj}_{\vec{u}}(\vec{v})$

projections (+.5)
start (+.5)
sense/style (+.5)
logic (+1)

(geometric version on next page)

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