A Guide to Writing Mathematics

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Introduction

This is a math class! Why are we writing?

There is a good chance that you have never written a paper in a math class before. So you might be wondering why writing is required in your math class now.

The Greek word *mathemas*, from which we derive the word mathematics, embodies the notions of knowledge, cognition, understanding, and perception. In the end, mathematics is about *ideas*. In math classes at the university level, the ideas and concepts encountered are more complex and sophisticated. The mathematics learned in college will include concepts which cannot be expressed using just equations and formulas. Putting *mathemas* on paper will require writing sentences and paragraphs in addition to the equations and formulas.

Mathematicians actually spend a great deal of time writing. If a mathematician wants to contribute to the greater body of mathematical knowledge, she must be able communicate her ideas in a way which is comprehensible to others. Thus, being able to write clearly is as important a mathematical skill as being able to solve equations. Mastering the ability to write clear mathematical explanations is important for non-mathematicians as well. As you continue taking math courses in college, you will come to know more mathematics than most other people. When you use your mathematical knowledge in the future, you may be required to explain your thinking process to another person (like your boss, a co-worker, or an elected official), and it will be quite likely that this other person will know less math than you do. Learning how to communicate mathematical ideas clearly can help you advance in your career.

You will find that writing good mathematical explanations will improve your knowledge and understanding of the mathematical ideas you encounter. Putting an idea on paper requires careful thought and attention. Hence, mathematics which is written clearly and carefully is more likely to be correct. The process of writing will help you learn and retain the concepts which you will be exploring in your math class.

What does good mathematical writing look like?

As you learn more math, being able to express mathematical ideas will become more important. It will no longer be sufficient just to be able to write down some final "answer". There is a good reason why Herman Melville wrote *Moby Dick* as a novel and not as the single sentence:

The whale wins.

For this same reason, just writing down your final conclusions in an assignment will not be sufficient for a university math class.

You should not confuse writing mathematics with "showing your work". You will not be writing math papers to demonstrate that you have done the homework. Rather, you will be writing to demonstrate how well you understand mathematical ideas and concepts. A list of calculations without any context or explanation demonstrates that you've spent some time doing computations; however, a list of calculations without any explanations omits ideas. The ideas are the mathematics. So a page of computations without any writing or explanation *contains no math.*

When you write a paper in a math class, your goal will be to communicate mathematical reasoning and ideas clearly to another person. The writing done in a math class is very similar to the writing done for other classes. Your are probably already used to writing papers in other subjects like psychology, history, and literature. You can follow many of the same guidelines in a mathematics paper as you would in a paper written about these other subjects.

Basics: Combining Words and Equations

Following the rules of grammar.

Good writing observes the rules of grammar. This applies to writing in mathematics papers as well! When you write in a math class, you are expected to use correct grammar and spelling. Your writing should be clear and professional. Do not use any irregular abbreviations or shorthand forms which do not conform to standard writing conventions. Mathematics is written with sentences in paragraphs. (And yes, paragraphs are important. It is not amusing to read a three-page paper consisting of just one paragraph.)

There is however one element in mathematical writing which is not found in other types of writing: formulas. However, it may surprise you to know that in a math paper, formulas and equations follow the standard grammatical rules that apply to words. Mathematical

symbols can correspond to different parts of speech. For instance, below is a perfectly good complete sentence.

$$1 + 1 = 2.$$

The symbol "=" acts like a verb. Below are a couple more examples of complete sentences.

$$3xy < -2.$$
$$5z \in \mathbb{R}.$$
$$9 - s \neq t.$$

Can you identify the verbs? On the other hand, an expression like

$$5x^2z - 10y$$

is not a complete sentence. There is no verb. Such an expression should be treated as a noun. Can you identify the nouns in the previous examples?

Formulas and equations need to be contained in complete sentences with proper punctuation. Here is an example:

> The total revenue, R, made from selling widgets is given by the equation
>
> $$R = pq,$$
>
> where p is the price at which each widget is sold and q is the number of widgets sold. Based on past experience, we know that when widgets are priced at \$15 each, 2000 widgets will be sold. We also know that for every dollar increase in price, 150 fewer widgets are sold. Hence, if the price is increased by x dollars, then the revenue is
>
> $$\begin{aligned} R &= (15 + x)(2000 - 150x) \\ &= -150x^2 - 250x + 30,000. \end{aligned}$$

Notice how punctuation follows each of equations. A computation which ends a sentence needs to end with a period. Computations which do not end sentences are followed by commas.

A good way to improve your mathematical writing is by reading your writing, *including all of the equations*, out loud. Your ears can often pick out sentence fragments and grammatical errors better than your eyes. If you find yourself saying a series of fragmented sentences and equations, you should do some rewriting.

There are a couple of other important things to observe in the above example. Notice how "we" is used. The use of first person is common in mathematics, especially the plural "we", so don't be afraid to use the word "we" in the papers you write in your math class.

Another thing to notice is that important or long formulas are written on separate lines. You can make your mathematical writing easier to read if you place each important formula on a line of its own. It's hard to pick out the important formulas below:

> If d is Bob's distance above the ground in feet, then $d = 100 - 16t^2$, where t is the number of seconds after Bob's Flugelputz-Levitator is activated. Solving for t in the equation $100 - 16t^2 = 0$, we find that $t = 2.5$. Bob hits the ground after 2.5 seconds.

This is clearer:

> If d is Bob's distance above the ground in feet, then
>
> $$d = 100 - 16t^2,$$
>
> where t is the number of seconds after Bob's Flugelputz-Levitator is activated. Solving for t in the equation
>
> $$100 - 16t^2 = 0,$$
>
> we find that $t = 2.5$. Bob hits the ground after 2.5 seconds.

Symbols and words.

It is important to use words and symbols appropriately. Part of being able to write mathematics well is knowing when to use symbols and knowing when to use words.

Don't use mathematical symbols when you really mean something else. A common mistake is to misuse the "=" symbol. For instance:

$$3^{2x} - 2(3^x) = -1 \;=\; (3^x)^2 - 2(3^x) + 1 = 0 \;=$$
$$(3^x - 1)^2 = 0 \;=\; 3^x = 1 \;=\; x = 0.$$

!!

Do not use the equal sign when you really mean "the next step is" or "implies". The above example is really saying that $-1 = 0 = 1$! Using arrows instead of equal signs is a slight improvement, but still not desirable:

$$3^{2x} - 2(3^x) = -1 \;\rightarrow\; (3^x)^2 - 2(3^x) + 1 = 0 \;\rightarrow\; (3^x - 1)^2 = 0 \;\rightarrow\; 3^x = 1 \;\rightarrow\; x = 0.$$

With a sequence of calculations, sometimes it is best to just place each equation on a separate line.

$$\begin{aligned} 3^{2x} - 2(3^x) &= -1 \\ (3^x)^2 - 2(3^x) + 1 &= 0 \\ (3^x - 1)^2 &= 0 \\ 3^x &= 1 \\ x &= 0. \end{aligned}$$

For a difficult computation where the reader might not readily follow each step, you can include words to describe the steps you take.

> We want to solve for x in the equation
>
> $$3^{2x} - 2(3^x) = -1.$$
>
> We can rewrite this equation in terms of 3^x:
>
> $$(3^x)^2 - 2(3^x) + 1 = 0.$$
>
> After factoring, this becomes
>
> $$(3^x - 1)^2 = 1$$
>
> and it follows that $3^x = 1$, or $x = 0$.

However, make sure that your paper has a single flow. Don't explain a calculation using the "two-column method".

$3^{2x} - 2(3^x) = -1$	Solve this equation.
$(3^x)^2 - 2(3^x) + 1 = 0$	Collect the terms on one side.
$(3^x - 1)^2 = 0$	Factor.
$3^x = 1$	Use the Zero Factor Property.
$x = 0$	Solve for x.

This is hard to read through. It's also bad style.

Some things are best expressed with words. But other things are best expressed with mathematical notation. For instance, it hard to read:

> It follows that x plus two is larger than zero.

Here, mathematical notation is more appropriate.

> It follows that $x + 2 > 0$.

Miscellaneous comments.

Here are a couple of other pointers to help you get started with your mathematical writing.

- Don't start a sentence with a formula. While it may be grammatically correct, it looks strange.

> $t = 5$ when $w = 2000$, so we can conclude that the new factory will be completely overrun with cockroaches in 5 years.

> f is globberfluxible at $x = 3$.

Adding just a word or two can fix these examples.

> Since $t = 5$ when $w = 2000$, we can conclude that the new factory will be completely overrun with cockroaches in 5 years.

> The function f is globberfluxible at $x = 3$.

- Don't turn in pages of unreadable scribbles to your professor. In college, papers are typed. They are also usually double-spaced with large margins. Mathematics papers adhere to the same standards as papers written for other classes.

- While it is a good idea to type your paper, you may have to leave out the formulas and insert them by hand later. It is perfectly acceptable to write formulas by hand in a math paper. Just make sure that your mathematical notation is legible. If you do decide to type the equations, please be aware that variables in equations and formulas are usually italicized (to set them apart from the text). Many word processing programs contain equation editors. In newer versions of Microsoft Word, the equation editor is available under the `Insert` menu. Select `Object...`, and then `Equation`.[1] If you are going to be writing a lot of technical documents, it might be worthwhile to learn TEX or LATEX. These are professional mathematical typesetting languages. This document was written with LATEX. You may also find satisfactory results typing papers in Maple or some other mathematically oriented software program.

- Use mathematical notation correctly. As you learn to write more complicated formulas, it is all too easy to leave out symbols from formulas. Learn how to use symbols properly!

- Use language precisely and correctly. Make sure that the words you use really mean what you think they mean. Mathematics requires very precise use of language. Another thing to avoid is overuse of the word "it". Mathematical papers with a lot of pronouns like "it" and "that" tend to be hard to read. It is often hard for the reader to see what "it" is referring to. If you, the author, are also having difficulty seeing what "it" is referring to, then you may be having some difficulty with the mathematical ideas; you may need to think more about the ideas you are writing about.

- Try to write as simply and directly as possible. No one likes to read ponderous pretentious prose.

[1] In Microsoft Word, it is also possible to place a button on the tool bar which activates the equation editor. Select `Configure...` beneath the `Tools` menu. In the window that pops up, select the `Commands` tab. Under the `Insert` category you will find the `Equation Editor` command. Drag the equation editor icon to the tool bar.

need to show everything.) The reader of a college mathematics paper will probably not be interested in reading how to multiply 5 and 74. Leave out what is unimportant. On the other hand, don't leave out anything which is critical to the key ideas you are trying to explain. Learning what is important and what is unimportant will help you understand mathematics better.

You should not assume that the reader is familiar with the problem you are solving. While you do not need to restate the problem in its entirety, be sure to give an overview of all important details in the problem. You also should not assume that the reader is in the same mind set as you. In your writing, state any assumptions which you have made. For instance, in physics problems, it is often assumed that everything is frictionless. But just because this assumption is made nearly all the time doesn't mean that your reader will automatically make this assumption; your reader may not be familiar with physics. Just because you assume something is true doesn't mean that your reader will. So write it down!

Defining variables and formulas.

Quantities and functions can be, and often should be, represented with letters. However, the letters which are chosen are arbitrary. You should explicitly state what all letters in your formulas represent in as precise a manner as possible. For instance:

Either n or $n+1$ is even.

What is n? If $n = 8.5$ is the above statement true? A better way of stating this is:

For any whole number n, either n or $n+1$ is even.

A common phrase used in mathematics is "Let...".

Let x be any real number.

Let P be the population of Los Angeles in 2010.

Let $f(x) = x^2 + 1$.

In the last example, x is a place holder. It doesn't require a proper introduction. However, it would be better to write:

> Let $f(x) = x^2 + 1$ for all real numbers x.

If describing all the variables gets tedious, try not assigning any variables at all. The following example clearly needs improvement.

> The volume is ℓwh.

The following example is adequate, but wordy.

> The volume of the box is ℓwh, where ℓ is the length, w is the width, and h is the height.

We can write this most elegantly by removing the variables.

> The volume of the box is the product of the length, the width, and the height.

!

You need to be especially careful with variables representing real-world quantities. Avoid describing them vaguely, as in:

> Let $D(t)$ be the distance at a time t.

Including units would make this clearer, but the description is still vague.

> Let $D(t)$ be the distance in miles at t hours.

Try to be as specific as possible.

> Let $D(t)$ be Agnes's distance from the arena in miles t hours after the riot began.

Also, be careful that each symbol you use represents only one thing. This can actually be more subtle than it sounds. The following example seems to be rather clear.

Let P be the escaped wombat population (in thousands) t years after 1990 and suppose that

$$P = 0.5(1.12)^t.$$

The wombat population in 1992 is approximately 672. We can see this by setting $t = 2$ and observing that

$$P = 0.5(1.12)^2 = 0.6272 \text{ thousand wombats.}$$

If we want to predict when the wombat population will reach 2000, we set $P = 2$ and solve for t using logarithms.

$$\begin{aligned} 2 &= 0.5(1.12)^t \\ \log 2 &= \log 0.5 + t \log 1.12 \\ t &= \frac{\log 2 - \log 0.5}{\log 1.12} \approx 12.23 \text{ years.} \end{aligned}$$

The wombat population will reach 2000 in the year 2002.

I think that the above example would be considered unobjectionable by most readers. It looks very clear and understandable. The variable P is always standing for the wombat population. However, notice that in the first paragraph, P is the wombat population in general. In the next paragraph, $P = 0.6272$, the wombat population in 1992. And in the last paragraph, $P = 2$. The meaning of P appears to be changing every time that it is used. In the first paragraph, P represents the population at *any* time. In the other instances, P represents the population at *one particular* time. The problem can be fixed omitting some variables and adding others.

Let P be the escaped wombat population (in thousands) t years after 1990 and suppose that

$$P = 0.5(1.12)^t.$$

By substituting 2 for t in the above equation, we can see that in 1992, the wombat population is approximately 672.

$$0.5(1.12)^2 = 0.6272 \text{ thousand wombats.}$$

Let t_{2000} be the year when the wombat population reaches 2000. Then,

$$\begin{aligned} 2 &= 0.5(1.12)^{t_{2000}} \\ \log 2 &= \log 0.5 + t_{2000} \log 1.12 \\ t_{2000} &= \frac{\log 2 - \log 0.5}{\log 1.12} \approx 12.23 \text{ years.} \end{aligned}$$

The wombat population will reach 2000 in the year 2002.

While in the above example, we can afford a little bit of sloppiness with the variables, in more complex problems, this could be a source of potential trouble. When a symbol is used to represent two different things (even, or perhaps especially, if those things are similar), the reader (and the writer!) can become confused. A symbol used in two different ways is not only confusing, but often results in incorrect mathematics!

Just as variables need to be introduced carefully, also be sure not to pull formulas out of thin air. Tell the reader how you get each formula or what each formula means. It's not very pleasant to get hit with formulas without any warning.

Using pictures in mathematics.

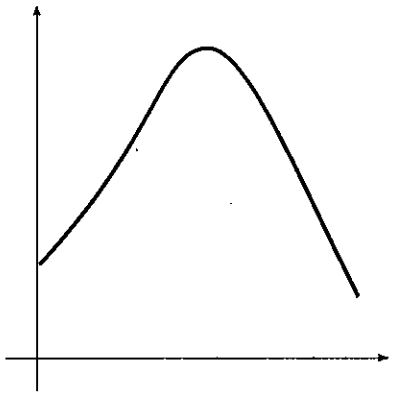
A picture can really be worth a thousand words. I strongly encourage you to use visual arguments in your mathematical writing. However, if you do include a picture, a diagram, a graph, or some other visual mathematical representation, make sure that you fully explain how it fits into your mathematical argument.

Looking at the graph, we can see that the result is true.

What should the reader look for in the graph? Why does the graph support the argument? Be more specific.

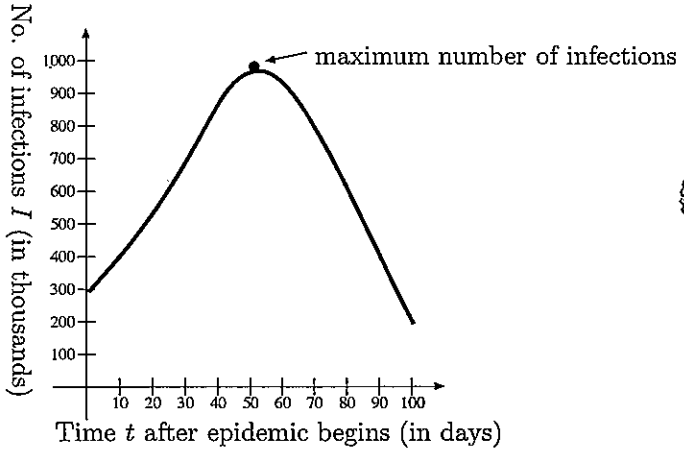
The graph increases sharply at $t = 3$, confirming our earlier prediction that the robots will begin a homicidal rampage three years from now.

A good graph should convey relevant and specific information to the reader. The following graph is vague.



Graphs and diagrams need to be neatly drawn and clearly labeled. Indicate the scale on the axes. You should point out significant graphical features.

Cooties infections versus time



If you draw a graph by hand, use a straight edge. You may want to generate your graphs using a computer. Be careful though. Programs like Excel or Microsoft Office generally are not good at generating mathematical graphs. You will more likely have success using a math program like Maple.

Any diagrams you draw should also be carefully labeled. Be sure to label everything that you refer to in your argument.