had 2 hours.

EXAM 2 — TMath 308 — Fall 2013

~~1000~~

NAME: Key

True/False: If the statement is *always* true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. [80] The vector $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the span $\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$.

(§2.3)

True +20

we need constants a, b & c so that $a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

ie is there a solution to

def +20 start +10 logic +20

$$\left[\begin{array}{ccc|c}1&0&1&1\\1&1&0&2\\0&1&1&3\end{array}\right] \xrightarrow{R_2-R_1\to R_2} \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&1&1&3\end{array}\right]$$

$$\xrightarrow{R_3-R_2\to R_3} \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&0&2&2\end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&1\\0&0&1&1\end{array}\right] \xrightarrow[R_1-R_3\to R_1]{R_2+R_3\to R_2} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&2\\0&0&1&1\end{array}\right]$$

one solution +10 computation setup

2. [80] Given matrices A, B, and C and that $A * B = A * C$, then $B = C$.

(§3.1)

False +20

start +10

This was a homework question! If A is not invertible we should be able to make this fly... let $A = \begin{bmatrix}1&1\\2&2\end{bmatrix}$ then

$$\begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}1&1\\-1&-1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} = \begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}2&2\\-2&-2\end{bmatrix}$$

looking for counterex +20
got one +10
mult +20

but $\begin{bmatrix}1&1\\-1&-1\end{bmatrix} \neq \begin{bmatrix}2&2\\-2&-2\end{bmatrix}$

3. [80] If A and B are both 2×2 matrices, then $(A + B)^{-1} = A^{-1} + B^{-1}$.

False +20

(§3.3 suggested problem)

+20 looking for counter ex
+10 got one

Consider $A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $B = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

start +10

+20 notice $A^{-1} = \frac{1}{(1)(1)-(0)(0)}\begin{bmatrix}1&-0\\-0&1\end{bmatrix}$ and $B^{-1} = \frac{1}{(-1)(-1)-(0)(0)}\begin{bmatrix}-1&-0\\-0&-1\end{bmatrix}$

so $A^{-1} + B^{-1} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ but $(A+B)^{-1} = \begin{bmatrix}0&0\\0&0\end{bmatrix}^{-1}$

which is not defined!

4. [80] The set $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$ is linearly independent.

(§2.3)

True +20

start +10

logic +20

def +20

A set is linearly independent if we can only write $\vec{0}$ as the 'trivial' linear combination. That is, there is only one solution to the homogeneous system of lin eq:

$a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ie $\left[\begin{array}{ccc|c}1&0&1&0\\1&1&0&0\\0&1&1&0\end{array}\right]$

work from #1 ⇒ the above's rref is $\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$

⇒ there is only one solution.. $a=0=b=c$

comp setup +10

5. [80] If A and B are symmetric $n \times n$ matrices, then AB is symmetric.

False +20

start +10

Recall A is symmetric if $A = A^T$

Consider $A = \begin{bmatrix}1&2\\2&-1\end{bmatrix}$ and $B = \begin{bmatrix}2&-1\\-1&0\end{bmatrix}$ (both symmetric)

notice $AB = \begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}2&-1\\-1&0\end{bmatrix} = \begin{bmatrix}0&-1\\3&-2\end{bmatrix}$

which is not symmetric

looking for counter ex +20

got one +10

mult +10

symmetric +10

6. [80] The set $\{(x, y, z) \in \mathbb{R}^3 | x - y + z = 1\}$ is a subspace.

False +20

start +10

def +20

Recall the three properties that define a subspace S

1) $\vec{0} \in S$

2) additive closure of S

3) scalar multiplicative closure of S

+20

In particular $[0,0,0] \notin \{(x,y,z) \in \mathbb{R}^3 \mid x-y+z=1\}$

b/c $0-0+0 \neq 1$

Notation/sense +10

Free Responce: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

7. Let A, B, and C be defined by:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -2 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute the following if possible.

(a) [60] $A + B^{\mathrm{T}}$

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 5 & 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 4 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}}_{+20}$$

start +10

got it +10

$$\underbrace{\begin{bmatrix} 7 & 0 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{+20}$$

(b) [40] CA

$(3\times3)(2\times3)$

start +10

can't be computed b/c the dimensions don't line upp +30

partial mult +10

(c) [60] a basis for col(A)

start +10

+10 $\text{Col}(A) = \text{span}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \subseteq \mathbb{R}^2$

There are lots of ways to find a basis ?

The way we talked about in class:

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 5 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 5 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2+R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} \end{bmatrix}$$

pivot col pivot col

So $\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}\right\}$ is a basis

lin indep +20

span +20

(d) [80] C^{-1}

Since I dont have a calculator..

$$\left[\begin{array}{ccc|ccc} -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

start +10

$$\xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

set up +30

$$\xrightarrow{R_2+R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

got it +10

algorithm +10

$$\xrightarrow{R_1+R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So $C^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

If wrong don't cut +10

if miss it -10 b/c should check

Check

$$\begin{bmatrix} -2 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ✓$$

8. [60] Given that A, B, and X are all matrices of appropriate sizes, solve for the matrix X. Assume all matrices are invertible and simplify as much as possible.

Solve for X (+10)
respect order of mult (+30)
Simplified (+20)

$$ABXA^{-1}B^{-1} = I + A$$

lots of correct answers?

$AB\,X\,A^{-1}\cancel{B^{-1}B} = (I+A)B$

$AB\,X\,\cancel{A^{-1}A} = (I+A)BA$

$\cancel{A^{-1}A}BX = A^{-1}(I+A)BA$

$\cancel{B^{-1}B}X = B^{-1}A^{-1}(I+A)BA$

$X = B^{-1}A^{-1}(I+A)BA$

$X = B^{-1}(A^{-1}I + \cancel{A^{-1}A})BA$

$X = B^{-1}(A^{-1}+I)BA$

$X = (B^{-1}A^{-1}+B^{-1}I)BA$

$X = (B^{-1}A^{-1}+B^{-1})BA$

$X = B^{-1}A^{-1}BA + \cancel{B^{-1}B}A$

$X = B^{-1}A^{-1}BA + A$

9. [80] Prove that if AB and BA are both defined then AB and BA are both square matrices.

Start (+10)
generality (+20)

Let A be an $m\times n$ matrix and B be a $p\times r$ matrix.

(+20)
Since $A\cdot B$ is defined we know $n=p$ by def. of matrix mult.
Since $B\cdot A$ is defined we know $r=m$ by def of matrix mult.

Thus B is really a $n\times m$ matrix.

sense/logic/notation (+10)

Since A is an $m\times n$ matrix & B is a $n\times m$ matrix we know by def of matrix mult that

(+20)
$A\cdot B$ is a $m\times m$ square matrix and that
$B\cdot A$ is a $n\times n$ square matrix. //

10. Let W be the set of all vectors $\overrightarrow{u} \in \mathbb{R}^2$ that exhibit the property:

$$||\overrightarrow{u} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}|| = ||\overrightarrow{u}|| + ||\begin{bmatrix} -1 \\ 1 \end{bmatrix}||.$$

In set builder notation:

$$W = \left\{ \overrightarrow{u} \in \mathbb{R}^2; ||\overrightarrow{u} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}|| = ||\overrightarrow{u}|| + ||\begin{bmatrix} -1 \\ 1 \end{bmatrix}|| \right\}.$$

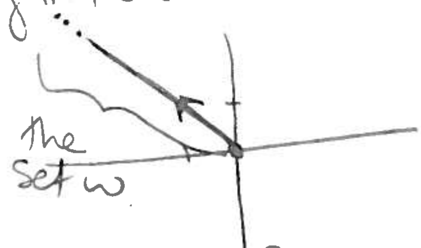
(a) [60] Describe geometrically what vectors the set W contains. (A picture is worth a thousand words and no proof is needed.)

(b) [80] Either prove W is a subspace or determine which subspace property is not satisfied by W.

a) This looks alot like that old homework problem: what vectors $\vec{u}$ and $\vec{v}$ exhibit $||\vec{u}+\vec{v}|| = ||\vec{u}|| + ||\vec{v}||$ only here $\vec{v}$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

start +10

The set W looks like all vectors pointing in the same direction as the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (and $\vec{0}$)



+30 if think line in correct direction
+10 if think line

b) Recall the three properties that define a subspace S
1) $\vec{0} \in S$ 2) additive closure of S 3) Scaler multiplicative closure
We'll consider each one

start +10 def +20

+10 1) ✓ notice that $\vec{0} \in W$ since
$\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \| = \| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \| = \sqrt{1+1} = \sqrt{2} = \| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \| + \| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \|$

~~eq~~ notation +20

+10 2) ✓ notice if $\vec{v}, \vec{u} \in W$ then by (a) we know $\exists\, c, d \in \mathbb{R}^+ \ni$ $\vec{v} = c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{u} = d\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so $\| (\vec{v} + \vec{u}) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \|$ will equal $\| (\vec{v} + \vec{u}) \| + \| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \|$ (all vectors are pointing in the same direction)

+30 3) Problem? Note $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \in W$, let $c = -1$ then $c\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which does not meet the condition to be in W