EXAM 1 — TMath 308 — Fall 2013

NAME: Key

True/False: If the statement is *always* true, give a *brief* explanation of why it is (not a formal proof!). If the statement is false, give a counterexample.

1. [4] Let $A = (3,4,5)$, $B = (2,4,2)$, $C = (-1,0,-3)$ be points in $\mathbb{R}^3$ and let O denote the origin. Then the vector $\overrightarrow{AB}$ is "the same" (i.e. equivalent to the) vector as ~~$\overrightarrow{OC}$~~ $\overrightarrow{CO}$.

start +.5
logic +1

$$\overrightarrow{AB} = \begin{bmatrix} 2-3 \\ 4-4 \\ 2-5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} \quad +.5 \qquad \overrightarrow{CO} = \begin{bmatrix} 0-{-1} \\ 0-0 \\ 0-{-3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad +.5$$

+1 FALSE in fact $\overrightarrow{AB} = -\overrightarrow{CO}$

finding vectors +.5

2. [4] For every vector $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^n$ and scalar $c \in \mathbb{R}$

$$c(\overrightarrow{u} + \overrightarrow{v}) = c\overrightarrow{u} + c\overrightarrow{v}.$$

+1 True. let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

start +.5
logic +1
general +.5

+.5 +.5

Then $c(\vec{u}+\vec{v}) = c\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = c\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$

$$= \begin{bmatrix} c(u_1+v_1) \\ c(u_2+v_2) \\ \vdots \\ c(u_n+v_n) \end{bmatrix} = \begin{bmatrix} cu_1+cv_1 \\ cu_2+cv_2 \\ \vdots \\ cu_n+cv_n \end{bmatrix} = c\vec{u} + c\vec{v} \checkmark$$

3. [4] The vectors $||\overrightarrow{u} + \overrightarrow{v}|| = ||\overrightarrow{u}|| + ||\overrightarrow{v}||$ if and only if $\overrightarrow{u} = \overrightarrow{0}$.

start +.5
logic +.5

If $\vec{u} = \vec{0}$ then $\|\vec{u}+\vec{v}\| = \|\vec{0}+\vec{v}\| = \|\vec{v}\| = \|\vec{v}\| + 0$
$= \|\vec{v}\| + \|\vec{u}\|$.

+1 False

However If $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ then by a homework problem we know

+1 $\{ \vec{u} \parallel \vec{v}$ or one of the vectors is zero.

Thus $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\| \not\Rightarrow \vec{u} = \vec{0}$.

4. [4] The following is the normal form of a line in $\mathbb{R}^3$, where $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is perpendicular to the line.

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \left(\begin{bmatrix}x\\y\\z\end{bmatrix} - \begin{bmatrix}3\\6\\9\end{bmatrix} \right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

stat +.5

logic +1.5

+1 FALSE. In $\mathbb{R}^3$ the 'normal form' of an equation defines a plane, not a line.

+1

AND.

vector doted with a vector returns a scaler

so $\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \left(\begin{bmatrix}x\\y\\z\end{bmatrix} - \begin{bmatrix}3\\6\\9\end{bmatrix} \right)$ must equal a scaler! This makes no sense!

+1

5. [4] The vector $\begin{bmatrix}5\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$

+1 True

stat +.5

logic +1

def +1: Are there real #'s s and t $\ni$

$s\begin{bmatrix}1\\-1\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}$?

ie $\begin{Bmatrix} s+t=5 \\ -s+t=1 \end{Bmatrix} \xrightarrow{R_1+R_2} 0s+2t=6 \Rightarrow t=3$

So $s+3=5 \Rightarrow s=2$.

Thus $\begin{bmatrix}5\\1\end{bmatrix} = 2\begin{bmatrix}1\\-1\end{bmatrix} + 3\begin{bmatrix}1\\1\end{bmatrix}$ ✓

ds +.5

6. [4] Every homogenous system of linear equations has at least one solution.

+1 True.

stat +.5

logic +1

def +1: A homogenous system of line. equations is one in which the constants are all equal to zero. This means of the form;

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0.$$

ds +.5: Note $x_1 = x_2 = \cdots = x_n = 0$ is a solution.

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Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

[7] 7. [X] The following problem is from the chapter on rectangular arrays in the Chiu Chang Suan Shu (Nine Chapters on the Mathematical Art) text written before 100CE. Solve it and be sure to show your work so I can follow your steps!

> The total yield of 3 sheaves of superior grain, 2 sheaves of medium grain, and 1 sheaf of inferior grain is 39 *duo* of rice. The total yield of 2 sheaves of superior grain, 3 sheaves of medium gran, and 1 sheaf of inferior grain is 34 *duo*. The total yield of 1 sheaf of superior grain, 2 sheaves of medium grain, and 3 sheaves of inferior grain is 26 *duo*. What is the yield of one sheaf of each grade of grain?

start (+.5)

(+1.5) translated into eqs

def variables consistent (+1)

let x be yield of superior grain/sheaf
let y be yield of medium grain/sheaf
let z be yield of inferior grain/sheaf

$$\begin{cases} 3x + 2y + z = 39 & \text{1st sentence} \\ 2x + 3y + z = 34 & \text{2nd sentence} \\ x + 2y + 3z = 26 & \text{3rd sentence} \end{cases}$$

we need to solve for x, y & z given

algorithm (+3)

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 2 & 3 & 1 & 34 \\ 3 & 2 & 1 & 39 \end{array}\right] \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 0 & -1 & -5 & -18 \\ 0 & -4 & -8 & -39 \end{array}\right] \xrightarrow[R_3 + 4R_2 \to R_3]{-R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 12 & 33 \end{array}\right] \xrightarrow[\frac{1}{12}R_3]{R_2 - 5R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 31/4 \\ 0 & 1 & 0 & 17/4 \\ 0 & 0 & 1 & 33/12 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 37/4 \\ 0 & 1 & 0 & 17/4 \\ 0 & 0 & 1 & 11/4 \end{array}\right]$$

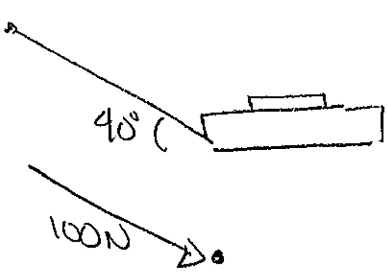
interpret correctly (+1.5)

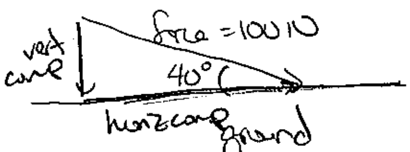
So the superior grain yields 37/4 duo
the medium grain gives 17/4 d
the inferior grain yields 11/4 duo

[3] 8. [X] A lawn mower has a mass of 30kg. It is being pushed with a force of 100N. If the handle of the lawn mower makes an angle of 40° with the ground, what is the ~~horizontal~~ horizontal component of the force that is causing the mower to move forward?

typo →

start (+.5) picture (+.5) looking for comp (+.5)





Sohcahtoa

$$\cos 40° = \frac{\text{horiz comp}}{100}$$ (+1)

$$\Rightarrow \text{horiz comp} = 100(\cos 40°)N$$ alg (.5)

$$\approx 76.6N$$

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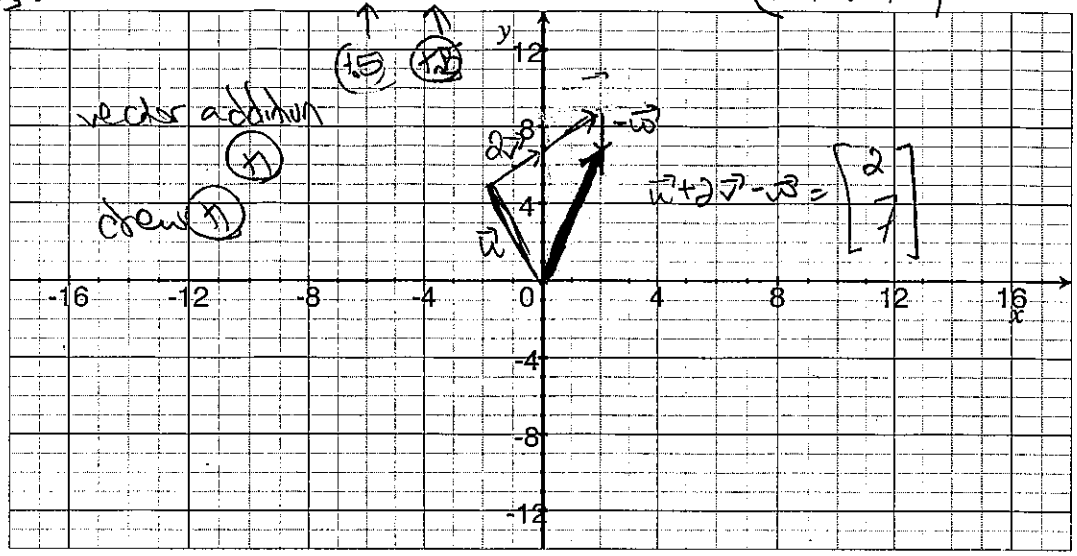
9. Let the following vectors be in $\mathbb{R}^2$.

$$\vec{u} = [-2, 5] \qquad \vec{v} = [2, 2] \qquad \vec{w} = [0, 2]$$

[3] (a) ~~[4]~~ Draw the vector $\vec{u} + 2\vec{v} - \vec{w}$ on the axes below. (bold vector)



[3] (b) ~~[4]~~ Find the angle between $\vec{u}$ and $\vec{v}$. Specify if you are using degrees or radians.

Recall $\underbrace{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\Theta}_{+.5}$ where Θ is the angle between $\vec{u}$ + $\vec{v}$.

$$\overbrace{(-2\cdot 2)+(5\cdot 2)}^{+1} = \overbrace{\sqrt{(-2)^2+(5)^2}\sqrt{2^2+2^2}}^{+1}\cos\Theta$$

$$\Rightarrow \arccos\left(\frac{6}{\sqrt{29}\sqrt{8}}\right) = \Theta \quad \text{deg } (+.5) \quad \Theta \approx 66.8^\circ$$

10. [5] Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the planes defined by $3x - 2y + z = -1$ and $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 5$ respectively. Write out the steps (done by hand) that you would take to find the line of intersection of the two planes. Note, you do *not* need to follow the steps! You just need to write them out so an intern (who took TMath308 before) could follow the steps.

steps +.5

+1.5 We need to find all (x,y,z) $\ni$ $\begin{Bmatrix} 3x-2y+z=-1 \\ 2x-y+4z=5 \end{Bmatrix}$

+1

1) Find the reduced row echelon form of: $\left[\begin{array}{ccc|c} 3 & -2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array}\right]$ (it was assumed the intern knows this)

2) Translate the matrix $\left[\begin{array}{ccc|c} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right]$ into $\begin{matrix} x+az=b \\ y+cz=d. \end{matrix}$

3) $\underbrace{\text{Let } z \text{ be the free variable}}_{+.5}$ + $\underbrace{\text{solve } x + y}_{+.5}$ $\Rightarrow$ $\begin{matrix} x = b-az \\ y = d-cz \end{matrix}$

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alive +1 4) The solution set is then $\left\{ \begin{bmatrix} b-ar \\ d-cr \\ r \end{bmatrix} \,\middle|\, r \in \mathbb{R} \right\}$ is the line.

11. [5] *Prove* $\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) = \vec{0}$ where $\vec{u}, \vec{v} \in \mathbb{R}^n$.

Algebra Pf We will use $\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}$ where $\vec{a}$ and $\vec{b}$ are in $\mathbb{R}^n$.

Start (+.5)

$\text{proj}_{\vec{u}}\vec{v}$ (+.5)

$\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}\vec{v})$ (+.5)

in general (+.5)

~~depe~~

sentences/explain (+1)

logic (+1)

We then compute $\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}(\vec{v}))$ using the properties of the dot produ

$$\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) = \text{proj}_{\vec{u}}\left(\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}\right)$$

$$= \left(\frac{\left[\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}\right]\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} \quad (+.5)$$

$$= \left[\frac{\left(\vec{v}\cdot\vec{u} - \left(\frac{\vec{u}\cdot\vec{v}}{\cancel{\vec{u}\cdot\vec{u}}}\right)\cancel{\vec{u}\cdot\vec{u}}\right)}{\vec{u}\cdot\vec{u}}\right]\vec{u}$$

$$= \left[\frac{\vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right]\vec{u}$$

$$= \left(\frac{0}{\vec{u}\cdot\vec{u}}\right)\vec{u} = 0\cdot\vec{u} = \vec{0}$$

Geometry Pf

Start (+.5)

$\text{proj}_{\vec{u}}\vec{v}$ described (+1)

in general (+.5)

sentences/explain (+1)

logic (+1.5) (cases etc)

Thus $\text{proj}_{\vec{u}}(\vec{v} - \text{proj}_{\vec{u}}(\vec{v})) = \vec{0}$. //