FINAL TQS 308 /50 Fall 2010

Key

True/False: If the statement is *always* true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. [4] Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

True Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

stated (+1)
linearity (+1)
logic (+1)
sense (+1)

Then $\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$ by comm. of $\mathbb{R}$

$= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \vec{v} + \vec{u}$. which is what we wanted to show.

2. [4] Let l be the line that passes through the point $(1, -1, 1)$ and has a direction vector $\vec{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. The line l is parallel to the plane defined by $2x + 3y - z = 1$.

stated (+1)
logic (+1)
computation (+1)
sense (+1)

$\ell = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ plane $\left.\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \right\}$ normal form?

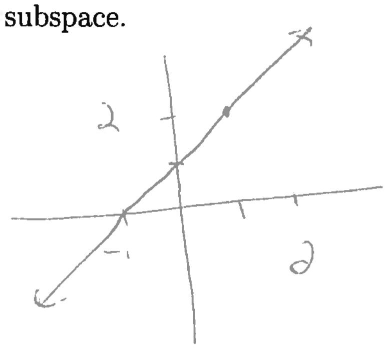
So the vector $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is normal to the plane

$\Rightarrow \ell$ is $\perp$ to the plane so false

3. [4] Any line in $\mathbb{R}^2$ is a subspace.

stated (+1)
def of subspace (+1)
logic (+1)
sense (+1)

false



is not a subspace b/c it does not contain the zero vector.

4. [4] A set of vectors that are linearly independent in $\mathbb{R}^3$ also form a basis for $\mathbb{R}^3$.

false. Consider $\{e_1, e_2\}$

The set is lin. indep, but does not form a basis for $\mathbb{R}^3$

stated (+1)
knew def of basis (+1)
def of lin indep (+1)
sense/logic (+1)

5. [4] For any matrices A and B, $\det(A+B) = \det(A) + \det(B)$.

false Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

then

$\det(A+B) = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ but $\det(A) + \det(B) = 0$

started (+1)
def of det (+1)
logic (+1)
sense (+1)

6. [4] All linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ have a nonzero eigenvector.

false Rotate the plane by $90°$

ie let the vector $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ act on $\mathbb{R}^2$.

Geometrically we can see no vectors retain their direction.

started (+1)
def of eigenvector (+1)
logic (+1)
sense (+1)

Free Responce: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.
(If you use a calculator, be sure to tell me.)

7. [3] Find all solutions to the following system of linear equations: $\left\{\begin{array}{c} x_1 - 3x_2 - 2x_3 = 0 \\ -x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 4x_2 + 6x_3 = 0 \end{array}\right\}$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow[R_3-2R_1 \to R_3]{R_2+R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 10 & 10 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+3R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $x + z = 0$, $y + z = 0$ $\Rightarrow$ if $z = s$ & is free, $x = -s$ & $y = -s$

So sol set $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} s \;\middle|\; s \in \mathbb{R} \right\}$

started (+.5)
calc (+1)
interpreted to x, y, z (+.5)
answered consistent (+1)

8. [2] Let $M = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 1 \\ 2 & 4 & 6 \end{bmatrix}$. Find a basis for Null($M$).
Hint: consider using your work from the previous question.

$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(+1) Knew def of Null + basis
(+1) used one from #7 correctly

9. [2] Let $A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$, and $C = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$. Determine if A, B, and C span $\mathbb{R}^3$.
Justify your result. *Hint: consider using your work from the previous questions.*

started (+.5) Nope, the rank of the matrix $[A\ B\ C]$ is
found thm of lin dep (+1) 2 thus C is a lin. combo of A & B
sense (+.5) $\Rightarrow \{A, B, C\}$ is not linearly indep
But $\{A, B\}$ is lin. indep but not enough vectors for $\mathbb{R}^3$ (who has dim 3).

10. Let $N = \begin{bmatrix} 0 & 0 & c \\ 0 & b & -c \\ a & a & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & a \\ 0 & f & a \end{bmatrix}$, and $Q = \begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$, where a, b, c, and f are nonzero real numbers. Find the following if possible:

(a) [1] $P + Q^{\mathrm{T}}$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & f & a \end{bmatrix} + \begin{bmatrix} 1 & 0 & f \\ 0 & f & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & a+f \\ 0 & 2f & a \end{bmatrix}$$

(b) [1] NP

3x3 2x3

not possible.

(a) [2] PQ

$$\begin{bmatrix} 1 & 0 & a \\ 0 & f & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & f \\ f & 0 \end{bmatrix}$$

2x3 3x2

$$\begin{bmatrix} 1+af & 0 \\ af & f^2 \end{bmatrix}$$

(b) [3] N^{-1}

$$\left[\begin{array}{ccc|ccc} 0 & 0 & c & 1 & 0 & 0 \\ 0 & b & -c & 0 & 1 & 0 \\ a & a & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{a}R_1]{R_1 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & b & -c & 0 & 1 & 0 \\ 0 & 0 & c & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_2 + R_3]{} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & b & 0 & 1 & 1 & 0 \\ 0 & 0 & c & 1 & 0 & 0 \end{array}\right]$$

$\frac{1}{b}R_2$
$\frac{1}{c}R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 1 & 0 & \frac{1}{b} & \frac{1}{b} & 0 \\ 0 & 0 & 1 & \frac{1}{c} & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{b} & -\frac{1}{b} & \frac{1}{a} \\ 0 & 1 & 0 & \frac{1}{b} & \frac{1}{b} & 0 \\ 0 & 0 & 1 & \frac{1}{c} & 0 & 0 \end{array}\right]$$

setup (+1)

algorithm (+1)

identified / got it (+1)

ense (+1)

esections (+1)

nerality (+1)

11. [4] Let $\vec{v}$ be a vector in $\mathbb{R}^n$, prove $||\vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$.

ic (+1) if $\vec{v} = \vec{0}$, we will show $||\vec{v}|| = 0$.

Recall by def. $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$ where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Then if $\vec{v} = \vec{0}$, $||\vec{0}|| = \sqrt{0^2 + 0^2 + \dots + 0^2} = \sqrt{0} = 0$ ✓

if $||\vec{v}|| = 0$, we will show $\vec{v} = 0$.

By def of $||\cdot||$, $||\vec{v}|| = 0 \Rightarrow \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = 0$

Square both sides $v_1^2 + v_2^2 + \dots + v_n^2 = 0$

Since $v_i^2 \geq 0$ for all i, the only way the sum can be zero is if each term is zero $\therefore v_i^2 = 0 \ \forall i$

$\Rightarrow v_i = 0$ for $i = 1, 2, \dots n$ $\Rightarrow \vec{v} = \vec{0}$.

12. Let k be a nonzero real number. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\vec{v} = A\vec{v}$, where $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.

(a) [3] Describe geometrically the effect of the matrix transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

test points (+1)

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} k \\ -1 \end{bmatrix}$$

All horizontal vectors remain untouched but vectors with any vert. component are stretched horiz.

(b) [1] What is the characteristic equation for A?

Knew def (+1)
got it (+1)

$Ax = \lambda x$

$Ax - \lambda x = 0$

$(A - I\lambda)x = 0$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} 1-\lambda & k \\ 0 & 1-\lambda \end{bmatrix} \rightarrow \det\left(\begin{bmatrix} 1-\lambda & k \\ 0 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(1-\lambda) - 0$$

(c) [2] Either use the above work or geometry to find the eigenvalues and an associated eigenvector for A.

Knew def (+1)
got one (+1)

Above b) eigenvalue is 1, so to find the eigenvectors

$$\left(\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & k & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow kx_2 = 0 \Rightarrow x_2 = 0$$

→ but x_1 can be anything.

Sol set: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} s \mid s \in \mathbb{R} \right\}$

(d) [2] Find the matrix that would record the following series of linear transformations on $\mathbb{R}^2$ with matrix multiplication:

i. apply the linear transformation T ← we know this matrix $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

ii. reflect over the y-axis. ← matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (+1)

So the two of them ii after i would be

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix}$$

note that if we reversed the order of the mult. we'd be wrong.

composition in right order (+.5)

mult (+.5)

~~9~~ 1

~~7~~ 31

~~11~~ 19

~~8~~ 50