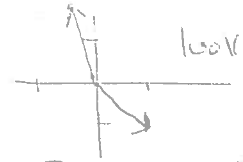
FINAL

TQS 308

Fall 2009

True/False: If the statement is *always* true, provide a proof of why it is. If the statement is false, give a counterexample.

1. [4] The angle between $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} \sqrt{3}-1 \\ \sqrt{3}+1 \end{bmatrix}$ is 60°.



looks obtuse

FALSE Recall $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \cos\theta$ for vectors $\vec{u}$ & $\vec{v}$ where θ is the angle between $\vec{u}$ & $\vec{v}$

$$\frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} \sqrt{3}-1 \\ \sqrt{3}+1 \end{bmatrix}}{\left\|\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\|\cdot\left\|\begin{bmatrix} \sqrt{3}-1 \\ \sqrt{3}+1 \end{bmatrix}\right\|} = \frac{(1)(\sqrt{3}-1)+(-1)(\sqrt{3}+1)}{\sqrt{(1)(1)+(-1)(-1)}\,\sqrt{(\sqrt{3}-1)^2+(\sqrt{3}+1)^2}} = \frac{-2}{\sqrt{2}\,\sqrt{3-\cancel{2\sqrt{3}}+1+3+\cancel{2\sqrt{3}}+1}}$$

$$= \frac{-2}{\sqrt{2}\sqrt{8}} = \frac{-2}{4} = \frac{-1}{2}$$

So $\cos\theta = -1/2 \Rightarrow \theta = 2\pi/3,\ 4\pi/3$ or coterminal angles

2. [4] For every vector $\overrightarrow{x}$ in $\mathbb{R}^n$ and scalar c, $||c\overrightarrow{x}|| = c||\overrightarrow{x}||$.

FALSE Let $\vec{x} \in \mathbb{R}^2$ and in particular $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let $c = -1$. Then

$$\|c\vec{x}\| = \left\|(-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\| = \sqrt{(-1)^2+0^2} = 1$$

but

$$c\|\vec{x}\| = (-1)\left\|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\| = (-1)\sqrt{1^2+0^2} = -1\cdot 1 = -1$$

and $1 \neq -1$

3. [4] If $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 4 & -2 & 1 & 3 \end{bmatrix}$, then $\begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$ is in Null(A).

$$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 4 & -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-2+0+0 \\ 4-4+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

TRUE Recall $\mathrm{Null}(A) = \{\vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0}\}$

4. [4] If $V = \text{Span}(\vec{v_1}, \vec{v_2}, \ldots \vec{v_n})$, then $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ forms a basis of V.

FALSE

Recall a basis is a set $\{\vec{b_1}, \vec{b_2}, \ldots \vec{b_i}\} \ni$ the set spans V and be a linearly independent set.

Consider $\text{Span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = V$. Notice that $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is not linearly independent b/c the 3rd vector is a sum of the first two

5. [4] The vector $\vec{x} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix}7 & 1 & -2\\-3 & 3 & 6\\2 & 2 & 2\end{bmatrix}$ with a corresponding eigenvalue of 6.

Recall a vector $\vec{v}$ is an eigenvector of A if $\exists\ \lambda \in \mathbb{R} \ni A\vec{v} = \lambda\vec{v}$.

$$A\vec{x} = \begin{bmatrix}7 & 1 & -2\\-3 & 3 & 6\\2 & 2 & 2\end{bmatrix}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-7+1+0\\3+3+0\\-2+2+0\end{bmatrix} = \begin{bmatrix}-6\\6\\0\end{bmatrix} = 6\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

So TRUE

6. [4] If $A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ then $A^n = \begin{bmatrix}2^{n-1} & 2^{n-1}\\2^{n-1} & 2^{n-1}\end{bmatrix}$ for all $n \geq 1$.

BC: If $n=1$ $A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ and $A^1 = \begin{bmatrix}2^0 & 2^0\\2^0 & 2^0\end{bmatrix}$ ✓

Induction. Assume $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}^k = \begin{bmatrix}2^{k-1} & 2^{k-1}\\2^{k-1} & 2^{k-1}\end{bmatrix}$ we want to consider

$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}^{k+1}$. So

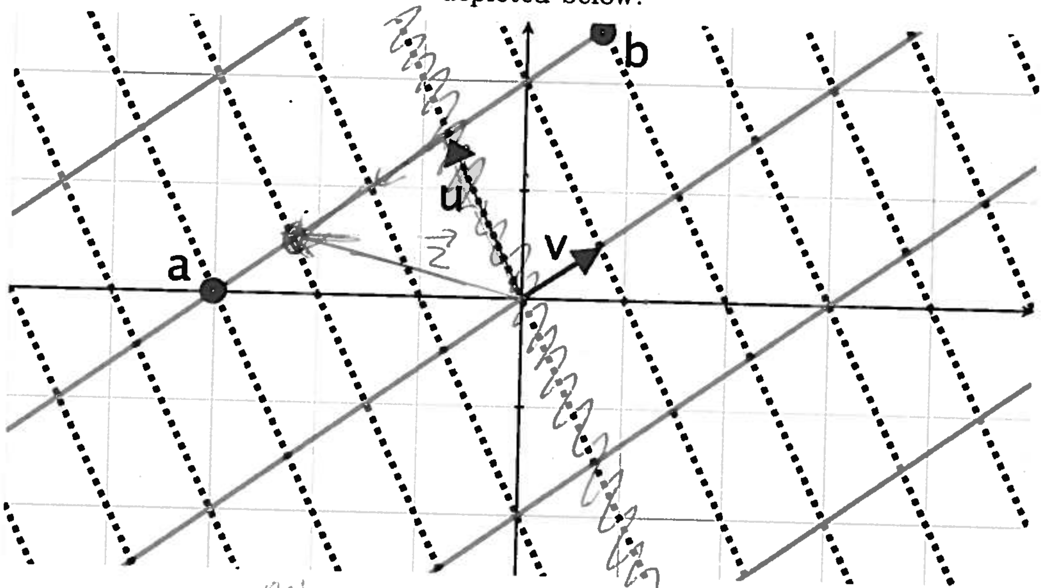
$$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}^{k+1} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}^k\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}2^{k-1} & 2^{k-1}\\2^{k-1} & 2^{k-1}\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}2^{k-1}+2^{k-1} & 2\cdot 2^{k-1}\\2^{k-1}+2^{k-1} & 2\cdot 2^{k-1}\end{bmatrix} = \begin{bmatrix}2^k & 2^k\\2^k & 2^k\end{bmatrix}$$

So TRUE by induction

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

1. Let $\vec{v}$ and $\vec{u}$ be vectors from $\mathbb{R}^2$ depicted below.



(a) [2] Label the ~~point~~ vector in $\mathbb{R}^2$ that corresponds to $\vec{u} - 2\vec{v} = \vec{z}$.

(b) [2] Let $U = \text{Span}(\vec{u})$. Identify U on $\mathbb{R}^2$ above.

the line with the squiggle on it

(c) [3] Write the vector form of the equation of the line that passes through a and b. Do not approximate $\vec{u}$ or $\vec{v}$ but use them if needed in the expression.

note $\vec{v}$ is the directional vector for the line through a & b
notice $\vec{u}$ provides a point on the line if the vector is in standard position so $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{u} + t\vec{v}$ where $t \in \mathbb{R}$

(d) [2] Notice that $\vec{u}$ and $\vec{v}$ span $\mathbb{R}^2$. Write $\vec{a}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

$\vec{a} = -\vec{v} - \vec{v} - \vec{v} + \vec{u} = -3\vec{v} + \vec{u}$

(e) [4] Is U a subspace? Justify your conclusion.

no, notice that $\vec{0}$ is not in $\vec{u}$.

2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a function that projects all the points in $\mathbb{R}^2$ onto the line $t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, where $t \in \mathbb{R}$.

(a) [4] Write down a matrix that acts on the point (x,y) in the same way that F does.

Recall that we can write down the matrix corresponding with F by tracing what F does to e_1 and e_2. The matrix would then be $\left[\vec{F(e_1)} \;\; \vec{F(e_2)} \right]$.

So $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(b) [3] Is F invertible? If so, find F^{-1}. If not, explain why.

No? Consider ~~the ...~~ $F\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $F\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Notice $F\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$

and $F\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - 1 \\ -1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$

Since both $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are sent to the same place the inverse can't 'undo' F.

3. Let $A = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

Note: you need to know how to take the determinant of a 3×3 matrix to do this problem.

(a) [4] Find the characteristic polynomial of A.

eigenvectors $\vec{x}$ are $\Rightarrow \exists\, \lambda \in \mathbb{R}$ with

$A\vec{x} = \lambda \vec{x}$

$\Rightarrow A\vec{x} - \lambda\vec{x} = \vec{0}$

$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

$\Leftrightarrow \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0$

$$\det\left(\begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -1-\lambda & 0 & 3 \\ 0 & -\lambda & 0 \\ 0 & 1 & -1-\lambda \end{bmatrix}\right)$$

$$= (-1-\lambda)\det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix}\right) - 0\cdot\det\left(\begin{bmatrix} 0 & 3 \\ 1 & -1-\lambda \end{bmatrix}\right) + 0\cdot\det\left(\begin{bmatrix} 0 & 3 \\ -\lambda & 0 \end{bmatrix}\right)$$

$$= (-1-\lambda)\left[(-\lambda)(-1-\lambda) - 1\cdot 0\right] = (-1-\lambda)(-\lambda)(-1-\lambda)$$

(b) [7] Find all the eigenvalues and a *basis* for their corresponding eigenspaces for the matrix A.

Eigenvalues correspond with values λ such that $\det(A - \lambda I) = 0$

i.e. $(-1-\lambda)(-\lambda)(-1-\lambda) = 0$

$\Rightarrow -1-\lambda = 0$ or $-\lambda = 0$ or $-1-\lambda = 0$

$\Rightarrow -1 = \lambda$ or $\lambda = 0$ or $\lambda = -1$

Eigenspace associated with $\lambda = -1$, i.e. $\text{Null}(A - (-1)I)$

$$= \text{Null}(A + I) = \text{Null}\left(\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\vec{x} = \vec{0} \right\}$$

So $\left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $3x_3 = 0$, $x_2 = 0$, no restriction on x_1

Thus $\text{Null}(A+I) = \left\{ \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\}$ basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Eigenspace associated with $\lambda = 0$ i.e. the Null space of A.

$\text{Null}(A - 0I) = \text{Null}(A) = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \}$

So $\left[\begin{array}{ccc|c} -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ x_3 is free, $x_1 - 3x_3 = 0$, $x_2 - x_3 = 0$ $\Rightarrow$ $x_1 = 3z$, $x_2 = z$ where $z \in \mathbb{R}$

basis $= \left\{ \begin{bmatrix} 3\cdot z \\ z \\ z \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$