Discovering and Proving that π Is Irrational

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Abstract. Ivan Niven's proof of the irrationality of π is often cited because it is brief and uses only calculus. However it is not well motivated. Using the concept that a quadratic function with the same symmetric properties as sine should when multiplied by sine and integrated obey upper and lower bounds for the integral, a contradiction is generated for rational candidate values of π. This simplifying concept yields a more motivated proof of the irrationality of π and π^2.

Charles Hermite proved that e is transcendental in 1873 using a polynomial that is the sum of derivatives of another polynomial [**7**]. Ivan Niven in 1947 found a way to use Hermite's technique to prove that π is irrational [**12**]. Lambert in 1767 had proven this result in a twelve-page article using continued fractions [**10**]. Niven's half-page proof, using only algebra and calculus, is frequently cited and sometimes reproduced in textbooks [**14, 9, 15, 4, 6**]. Although his proof is brief and uses ostensibly simple mathematics, it begins by defining functions as in the technique of Hermite without any motivation. In this article a simplifying concept is used that provides a more motivated and straightforward proof than Niven's. Using this concept, we, as it were, discover that π might be irrational and then confirm that it is with a proof.

1. A MOTIVATED APPROACH. We seek to combine a known falsity with a known truth and then to derive a contradiction from the combination. If π is assumed to be rational, $\pi = p/q$ with p and q natural numbers, then the maximum of $\sin x$ occurs at $p/2q$. The quadratic $-qx^2 + px = x(p - qx)$ will have its maximum at the same point, as will the product of the two functions. If we have a blender that allows inferences from this statement we might be able to derive a contradiction.

Such a blender exists in a definite integral. A definite integral allows for evaluations that might contradict upper or lower bounds. We have

$$0 < \int_0^{p/q} x(p - qx)\sin x\,dx \leq \frac{p^2}{4q} \cdot \frac{p}{q} = \frac{p^3}{4q^2}, \tag{1}$$

where the lower bound holds as the integrand is always positive,[1] and the upper bound is formed from the length of the interval of integration multiplied by the maximum value of the integrand [**16**, Property 8, p. 389].

For a polynomial $f(x)$, repeated integration by parts[2] gives the indefinite integral pattern

$$\int f(x)\sin x\,dx = -f(x)\cos x - f'(x)\sin x + f''(x)\cos x + f'''(x)\sin x - \cdots. \tag{2}$$

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[1] To see that the inequality is strict, consider:

$$\int_0^{p/4q} x(p-qx)\sin x\,dx + \int_{p/4q}^{3p/4q} x(p-qx)\sin x\,dx + \int_{3p/4q}^{p/q} x(p-qx)\sin x\,dx.$$

[2] Tabular integration by parts (see [**11**, p. 532] and [**5**]) is especially well suited for integrals of the type given in (1).

For the function $f(x) = x(p - qx)$, as $f^{(k)}(x) = 0$ for $k \geq 3$, we have

$$\int_0^{p/q} f(x) \sin x \, dx = \{-f(x) \cos x - f'(x) \sin x + f''(x) \cos x\}\Big|_0^{p/q} \tag{3}$$

and the odd term drops out ($\sin p/q = \sin 0 = 0$) leaving an alternating sum of even derivatives of $f(x)$ evaluated at the endpoints:

$$\int_0^{p/q} f(x) \sin x \, dx = f(p/q) + f(0) - f''(p/q) - f''(0). \tag{4}$$

This sum is $4q$. Combining (1) and (4) we have

$$0 < 4q \leq \frac{p^3}{4q^2}. \tag{5}$$

2. DISCOVERING π IS IRRATIONAL.

2.1. Candidate π Values. The inequalities in (5) show π does not equal 1 or 2. For $\pi = 7/2$, this $n = 1$ case of the general polynomial $x^n(p - qx)^n$ does not give a contradiction. We will try the $n = 2$ case and see if it works for this rational. This is possible as the same reasoning about $x(p - qx)$ applies to $x^n(p - qx)^n$: it is symmetric like $\sin x$ on $[0, p/q]$ and $x^n(p - qx)^n \sin x$ when integrated in that interval should have a value consistent with the integral's upper and lower bounds.

2.2. The $n = 2$ Case. With $f(x) = x^2(p - qx)^2$, repeated integration by parts gives

$$\int_0^{p/q} f(x) \sin x \, dx = f^{(0)}(p/q, 0) - f^{(2)}(p/q, 0) + f^{(4)}(p/q, 0), \tag{6}$$

where $f^{(k)}(p/q, 0) = f^{(k)}(p/q) + f^{(k)}(0)$. Multiplying out $f(x)$, we have

$$f(x) = x^2(p - qx)^2 = q^2x^4 - 2pqx^3 + p^2x^2. \tag{7}$$

Derivatives for this function are easily computed. The values of these derivatives at the endpoints 0 and p/q are given in Table 1.

Table 1. Derivatives of $x^2(p - qx)^2$.

k	$f^{(k)}(0)$	$f^{(k)}(p/q)$
0	0	0
1	0	0
2	$2! \cdot p^2$	$2! \cdot p^2$
3	$-3! \cdot 2pq$	$3! \cdot 2pq$
4	$4! \cdot q^2$	$4! \cdot q^2$

Using Table 1, with the same logic used for the inequalities in (5), we form the inequality

$$0 < -4p^2 + 48q^2 \leq \frac{p}{q}\left(\frac{p^2}{4q}\right)^2 \tag{8}$$

and letting $p = 7$ and $q = 2$ we get $-4p^2 + 48q^2 = -4$, a contradiction of the lower bound.

2.3. The $n = 3, 4$ Cases. Similar calculations can be carried out for the $n = 3$ and $n = 4$ cases. The inequalities for each are

$$0 < -144p^2q + 1440q^3 \le \frac{p}{q}\left(\frac{p^2}{4q^2}\right)^3 \tag{9}$$

and

$$0 < 48p^4 - 8640p^2q^2 + 80640q^4 \le \frac{p}{q}\left(\frac{p^2}{4q}\right)^4, \tag{10}$$

respectively.[3]

For the $n = 3$ case, when p/q equals 3/1, 13/4, 16/5, and 19/6 the upper or lower bound of (9) is contradicted. We discover that 22/7 is not π using (10), the $n = 4$ case.

We have evidence that our method can be used to prove π is irrational.

3. PROVING π IS IRRATIONAL.

3.1. The General Case. Referring to Table 1, it is likely that $f(x) = x^n(p - qx)^n$ will be such that the alternating sum of its even derivatives evaluated at the endpoints 0 and p/q will be divisible by $n!$. If the integral in

$$0 < \int_0^{p/q} x^n(p - qx)^n \sin x \, dx \le \frac{p}{q}\left(\frac{p^2}{4q^2}\right)^n < p^{2n+1} \tag{11}$$

is divisible by $n!$, then the upper bound in (11) can be used to prove π is irrational. This follows as the integral is increasing with n factorially, but the upper bound has polynomial growth. We know factorial growth exceeds polynomial—see [**16**, Equation 10, p. 764]; [**3**, Example 2, p. 86] gives a direct proof of this result.

3.2. Proving the General Case. The lower and upper bounds of (11) follow from the properties of the integrand. Repeated integration by parts establishes that

$$\int_0^{p/q} x^n(p - qx)^n \sin x \, dx = \sum_{k=0}^{n} (-1)^k f^{(2k)}(p/q, 0). \tag{12}$$

Consequently, we need only prove that the right-hand side of (12) is divisible by $n!$.

First, symmetry of $f(x)$ allows us to consider only the left endpoint in this sum. This follows as the equation $f(x) = f(p/q - x)$, differentiated repeatedly, gives $f'(x) = -f'(p/q - x)$, $f''(x) = f''(p/q - x)$, and, by induction, $f^{(k)}(x) = (-1)^k f^{(k)}(p/q - x)$. So $f^{(k)}(0) = (-1)^k f^{(k)}(p/q)$. For the even derivatives, with which we are concerned, we have $f^{(2k)}(0) = f^{(2k)}(p/q)$.

[3]Leibniz's formula [**1**, Problem 4, p. 222] gives a means of calculating nth derivatives of a product of two functions. In the case of the product of two polynomials, all derivatives can be calculated by placing the derivatives of one polynomial along the top row of a table, the derivatives of the other polynomial along the left column, and forming a Pascal's triangle in the interior of the table. After forming products of these row and column entries with the binomial coefficients of Pascal's triangle, all derivatives are given by sums along interior diagonals of the table.

Next, $f(x)$ when expanded will have the form $a_n x^{2n} + \cdots + a_0 x^n$. For $k < n$, $f^{(k)}(0) = 0$, and for $k \geq n$, $f^{(k)}(0)$ is divisible by $k!$ and therefore $n!$. We have established that the sum in (12) is divisible by $n!$ and that π must be irrational.

4. CONCLUSION. Niven gives two proofs of the irrationality of π. One has been cited in the introduction. The other occurs in his book on irrational numbers [**13**]; there he shows the irrationality of π^2. We will re-examine these proofs.

Looking at Hermite's transcendence of e proof [**8**, p. 152], one sees definitions of two functions $f(x)$ and $F(x)$ with the derivatives of $f(x)$ being used in the definition of $F(x)$. An integral is then used with the integrand having e^{-x} in it. In Niven's π and π^2 proofs he defines one function as the sum of derivatives of the other, as Hermite does. The manipulations Niven performs are to obtain forms like Hermite's. In both articles the integral of one function equals an expression involving the other. To someone unsteeped in Hermite's technique the motivation for the proof must be unclear.

In this note a concept motivates the introduction of the polynomial Niven defines. The concept is that if π is rational then the evaluation of a definite integral comprised of the product of two functions symmetric about $x = \pi/2$ should be consistent with bounds for the integral. This being shown not to be the case, a contradiction occurs and π is proven irrational. The graphs of $\sin x$, $x(p - qx)$, and their product give the concept—visually.

The same logic used for π can be applied to π^2. Assume $a/b = \pi^2$. We have

$$0 < \int_0^{a/b} x^n (a - bx)^n \sin \frac{x}{\sqrt{a/b}}\, dx \leq \frac{a}{b} \left(\frac{a^2}{4b} \right)^n, \tag{13}$$

with the same reasoning as before: the integrand by assumption is a symmetric function with its maximum at $x = a/2b$. The integral, using repeated integration by parts, evaluates to

$$\sum_{k=0}^{n} (-1)^k (\sqrt{a/b})^{2k+1} (f^{(2k)}(p/q) + f^{(2k)}(0)) \tag{14}$$

where $f(x) = x^n (a - bx)^n$. With some factoring, this sum is

$$\frac{\pi}{b^n} \sum_{k=0}^{n} (-1)^k b^{n-k} a^k (f^{(2k)}(p/q) + f^{(2k)}(0)). \tag{15}$$

With a multiplication by b^n/π to clear π/b^n from this sum, we have then

$$0 < \frac{b^n}{\pi} \int_0^{a/b} x^n (a - bx)^n \sin \frac{x}{\sqrt{a/b}}\, dx = n! R_n \leq \frac{b^n}{\pi} \frac{a}{b} \left(\frac{a^2}{4b} \right)^n < a^{3n+1}, \tag{16}$$

which gives a contradiction.

Note: reproductions of older articles by Hermite [**8**] and others can be found in [**2**].

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