

NAME:

*Key*

True/False: If the statement is *always* true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. [4] If  $A$  and  $B$  are both  $2 \times 2$  matrices, then  $AB = BA$

(+) *False*

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

*but*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

2. [4] A matrix of the form  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$  where  $a, b \in \mathbb{R}$  commutes with  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  *multiplicatively*

True (+)  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

*logic (+)  
start (+)  
finish (+)  
shown in general (+)*

3. [4] If  $A$  and  $B$  are both  $2 \times 2$  matrices, then  $(A+B)^{-1} = A^{-1} + B^{-1}$ .

(+) *False* let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

*logic (+)  
start (+)  
counterexample (+)*

then  $(A+B)^{-1} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}^{-1} = \frac{1}{9-0} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

$A^{-1} + B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{2} & 0 \\ 0 & 1+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$

$\Rightarrow (A+B)^{-1} \neq A^{-1} + B^{-1}$

4. [4] A basis for the subspace:  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + r \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \mid r, t \in \mathbb{R} \right\}$  in  $\mathbb{R}^3$  is:  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$

$\textcircled{1}$  false 1) the set  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + r \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \mid r, t \in \mathbb{R} \right\}$  is  
 not a subspace  
 (bc it does not contain  $\vec{0}$ )  
 2) The set  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$  is not linearly indep  
 + thus not a basis.

5. [4] A set of vectors that are linearly independent in  $\mathbb{R}^3$  also form a basis for  $\mathbb{R}^3$ .

$\textcircled{1}$  false, A basis must contain a set that  
 is linearly independent and  
 spans  $\mathbb{R}^3$ .

For example  $\{e_1, e_2\}$  is a set of linearly indep  
 vectors, but it does not span  $\mathbb{R}^3$  + thus  
 is not a basis.

6. [4] The transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  that embeds  $\mathbb{R}^2$  into the  $xy$ -plane is given by

$$\vec{x} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}.$$

$$: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$\textcircled{1}$  false: the linear transformation has a domain  
 of  $\mathbb{R}^3$  + a range of  $\mathbb{R}^2$ .

Furthermore it projects  $\mathbb{R}^3$  onto the  
 x-y plane.

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

7. Let  $A$ ,  $B$ , and  $C$  be defined by:

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 4 & -4 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 \\ 6 \\ 9 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -2 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Compute the following if possible.

- (a) [1]  $2B$

$$2 \begin{bmatrix} -3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \\ 18 \end{bmatrix}$$

(+1)

- (b) [2]  $A^T$

$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \\ 0 & 1 \end{bmatrix}$$

size (+1)  
got it (+1)

- (c) [2] a basis for  $\text{col}(A)$

$$\begin{bmatrix} -2 & 2 & 0 \\ 4 & -4 & 1 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so 1<sup>st</sup> + 3<sup>rd</sup> col.

$$\left\{ \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

alg (+1)  
got it (+1)

- (d) [4]  $C^{-1}$

$$\left[ \begin{array}{ccc|ccc} -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

size (+1)

$$\begin{aligned} &\xrightarrow{-\frac{1}{2}R_1} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \quad \text{alg (+1)} \\ &\xrightarrow{R_2+R_3 \rightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_1+R_2 \rightarrow R_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \quad \text{got it / checke} \\ &\quad \text{(+1)}$$

8. Let  $A$  be a  $2 \times 3$  matrix,  $B$  be a  $3 \times 1$  matrix, and let  $C$  be a  $3 \times 3$  matrix.

- (a) [1] What are the dimensions of the matrix  $BC$ , if it exists?

$$3 \times 1 \quad 3 \times 3$$

$BC$  is not defined (+1)

- (b) [1] What are the dimensions of the matrix  $CAT^T$ , if it exists?

$$3 \times 3 \quad 3 \times 2$$

$3 \times 2$  (+1)

- (c) [1] What are the dimensions of the matrix  $C^8B$ , if it exists?

$$C \text{ is } 3 \times 3$$

$$3 \times 3 \quad 3 \times 1$$

$3 \times 1$  (+1)

$$C^2 \text{ is } 3 \times 3$$

3

$$C^8 \text{ is } 3 \times 3$$

9. Assume all matrices  $A_i$  given below are multiplicatively invertible and of such a size that the multiplication makes sense.

- (a) [2] What is the multiplicative inverse of  $A_2A_1$ ?

$$A_1^{-1}A_2^{-1} \text{ b/c } (A_2A_1)(A_1^{-1}A_2^{-1}) = A_2I A_2^{-1} = I$$

showed ① if didn't show it here check below. and similarly  $A_1^{-1}A_2^{-1}A_2A_1 = I$

- (b) [4] Use induction to prove that  $A_n \dots A_2A_1$  is invertible for all  $n$ .

To show  $A_n \dots A_2A_1$  is invertible we exhibit the inverse, namely  $A_1^{-1}A_2^{-1} \dots A_n^{-1}$ . We prove  $A_1^{-1}A_2^{-1} \dots A_n^{-1}$  is the inverse to  $A_n \dots A_2A_1$  by induction.

① The base case was shown above.

Induction  
sum ① We assume the inverse to  $A_{n-1} \dots A_2A_1$  is  $A_1^{-1}A_2^{-1} \dots A_{n-1}^{-1}$  and work to show that the inverse to  $A_n \dots A_2A_1$  is  $A_1^{-1}A_2^{-1} \dots A_n^{-1}$ .

assumption  
noticing ① logic ① Notice

$$(A_1^{-1}A_2^{-1} \dots A_{n-1}^{-1}A_n)(A_n \dots A_2A_1)$$

$$= (A_1^{-1}A_2^{-1} \dots A_{n-1}^{-1})(A_n^{-1}A_n)(A_n \dots A_2A_1)$$

$$= A_1^{-1}A_2^{-1} \dots A_{n-1}^{-1} I A_n \dots A_2A_1$$

$$= I \text{ by the inductive assumption}$$

$$\text{Similarly } (A_n \dots A_2A_1)(A_1^{-1}A_2^{-1} \dots A_{n-1}^{-1}A_n)$$

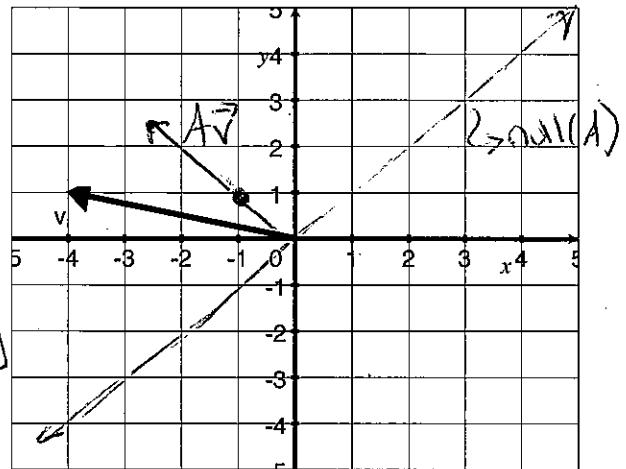
$$= A_n(A_n \dots A_2A_1A_1^{-1}A_2^{-1} \dots A_{n-1}^{-1}A_n)A_n = A_n I A_n^{-1} = I$$

10. Consider the transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $T\vec{v} = A\vec{v}$ , where

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

- (a) [1] The vector shown on the right is  $\vec{v}$ . Draw the image of  $\vec{v}$  under the transformation  $T$ .

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 - \frac{1}{2} \\ 0 + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{1}{2} \end{bmatrix} + 1 \quad \text{+1}$$



- (b) [3] Find a basis for  $\text{null}(A)$ .

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 0 & 0 & | & 0 \\ \frac{1}{2} & -\frac{1}{2} & | & 0 \end{bmatrix} \xrightarrow{2R_1 \rightarrow R_1} \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

aug (+1)  
got  $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$   
Knew what looking for (+1)  
R2/R1  $\rightarrow$  rotation 90°

Set  $s$  s.t.  $1 \cdot s - 1 \cdot y = 0$   
Then  $x - s = 0$   
 $\Rightarrow$  sol's are  $\left\{ \begin{bmatrix} s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\} \Rightarrow$  basis for  $\text{null}(A) : \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

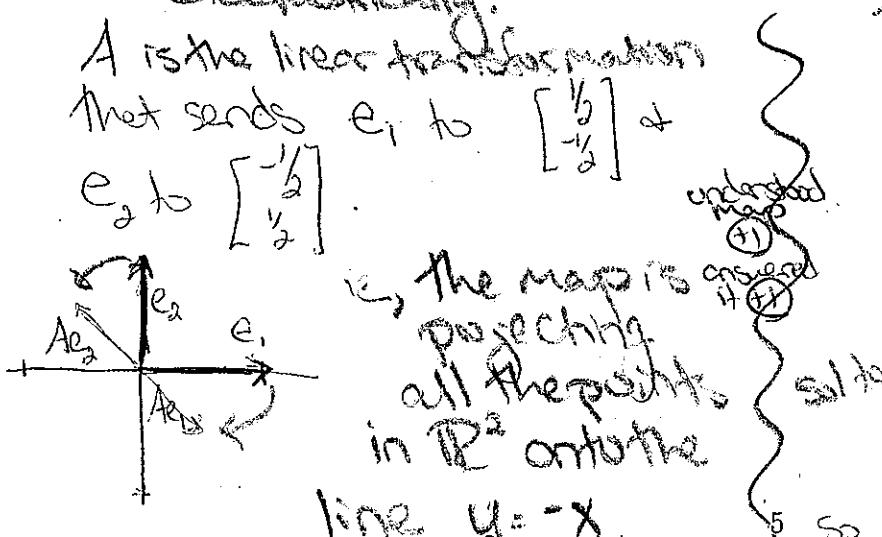
- (c) [2] Identify  $\text{null}(A)$  on the axes provided above.

only check for consistency (+2)

- (d) [2] Find any  $\vec{x}$  that are invariant under the action of  $T$ . That is, find all  $\vec{x}$  such that  $T\vec{x} = \vec{x}$ . Justify your answer either geometrically or algebraically.

Geometrically:

$A$  is the linear transformation  
that sends  $e_1$  to  $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$  +  
 $e_2$  to  $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ .



the map is causing  
projecting  
all the points  
in  $\mathbb{R}^2$  onto the

line  $y = -x$ .

so, the line  $y = x$  will  
remain invariant.

Algebraically: looking for all  $\begin{bmatrix} x \\ y \end{bmatrix}$   
so that  
set up for  
solved (+5)  
got it (+5)

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} \text{So } \frac{1}{2}x - \frac{1}{2}y &= x \quad \left\{ \begin{array}{l} \frac{1}{2}x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + \frac{1}{2}y = y \end{array} \right. \\ -\frac{1}{2}x + \frac{1}{2}y &= y \quad \left\{ \begin{array}{l} \frac{1}{2}x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x + \frac{1}{2}y = y \end{array} \right. \end{aligned}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so all  $x + y$  of the form that  
 $x + y = 0$  so all  $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$   
on orientation on the off-diagonal

~~24~~  
~~12~~  
~~6~~  
~~3~~

~~20~~  
~~30~~  

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~~50~~