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EXAM 1

TQS 308
Cdo

Fall 2010

25 min

NAME: Key

True/False: If the statement is *always* true, give a brief explanation of why it is. If the statement is false, give a counterexample.

1. [4] For every vector \vec{u} , \vec{v} and \vec{w} in \mathbb{R}^n

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}).$$

logic (+) start (+) counterex (+) write in real (+) True Let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ & $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$. Then the
LHS = $(\vec{u} + \vec{v}) + \vec{w} = \left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right) + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} (u_1 + v_1) + w_1 \\ \vdots \\ (u_n + v_n) + w_n \end{bmatrix}$
and the RHS = $\vec{u} + (\vec{v} + \vec{w}) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix} = \begin{bmatrix} u_1 + (v_1 + w_1) \\ \vdots \\ u_n + (v_n + w_n) \end{bmatrix}$

Thus by the associative property of the real #'s the LHS = RHS.

2. [4] For every vector \vec{u} , \vec{v} , and scalar c

$$c \cdot (\vec{u} + \vec{v}) = c \cdot \vec{u} + c \cdot \vec{v}.$$

(+) False,

It doesn't make sense to dot a scalar with a vector.

3. [4] The vectors $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u} = \vec{0}$.

logic (+) start (+) counterex (+) False. If $\vec{v} \neq \vec{0}$ but maybe $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
then LHS = $\|\vec{u} + \vec{v}\| = \left\| \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\| = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$
and the RHS = $\|\vec{u}\| + \|\vec{v}\| = \left\| \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \sqrt{4+4} + \sqrt{1+1}$
 $= \sqrt{8} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = \sqrt{2}(2+1) = 3\sqrt{2}$

but $\vec{u} \neq \vec{0}$.

4. [4] The following equation defines a plane in \mathbb{R}^3 :

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + r \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

(+) False, this is a line b/c $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(+) started
control
logic (+)

5. [4] Assume you are given a matrix with at least two rows. Perform $R_2 + R_1$ and $R_1 + R_2$. Now rows 1 and 2 are identical. Now perform $R_2 - R_1$ to obtain a row of zeros in the second row. Thus all matrices with at least two rows are row equivalent to a matrix with a zero row.

(+) False consider $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and follow the steps.

(+) started
logic (+)
control (+)
 $R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$ then $R_1 + R_2 \rightarrow \begin{bmatrix} 5 & 9 \\ 4 & 6 \end{bmatrix}$, the rows are not identical

6. [4] The span of

$$\begin{matrix} \text{"}e_1\text{"} & \text{"}e_2\text{"} & \text{"}e_3\text{"} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & \text{and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{matrix}$$

is all of \mathbb{R}^3 .

(+) True. Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$

then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

thus $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a linear combo of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
ie $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{span}\{e_1, e_2, e_3\}$

(+) started
knew span
logic (+)

Free Response: Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

7. [8] Use Gaussian or Gauss-Jordan elimination to find solution(s) if they exist, to the following system of linear equations: $\begin{cases} -2x_1 + 2x_2 = -10 \\ 6x_1 + 2x_2 + 2x_3 = 14 \end{cases}$

$$\begin{aligned} & \left[\begin{array}{ccc|c} -2 & 2 & 0 & -10 \\ 6 & 2 & 2 & 14 \end{array} \right] \xrightarrow{\substack{-\frac{1}{2}R_1 \\ \frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 3 & 1 & 1 & 7 \end{array} \right] \xrightarrow{R_2 - 3R_1 \rightarrow R_2} \\ & \left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 4 & 1 & -8 \end{array} \right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 1 & \frac{1}{4} & -2 \end{array} \right] \xrightarrow{R_1 + R_2 \rightarrow R_1} \\ & \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 3 \\ 0 & 1 & \frac{1}{4} & -2 \end{array} \right] \end{aligned}$$

Z-15

algorithm (+1) reduced form (+1) in x form (+1)

so $x_1 + \frac{1}{4}x_3 = 3$
 $x_2 + \frac{1}{4}x_3 = -2$
 \Rightarrow we have a free variable?

let $x_3 = s$ then
 $x_1 = 3 - \frac{1}{4}s$
 $x_2 = -2 - \frac{1}{4}s$
 and
 so the solutions are $\left\{ \begin{bmatrix} 3 - \frac{1}{4}s \\ -2 - \frac{1}{4}s \\ s \end{bmatrix} \mid s \in \mathbb{R} \right\}$

write sol correctly (+2)

8. [4] Find a reduced row echelon (or row echelon) form for A where $A = \begin{bmatrix} -2 & 2 & 0 & -10 \\ 6 & 2 & 2 & 14 \end{bmatrix}$

notice this is the same matrix as above thus

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 4 & 1 & -8 \end{array} \right] \text{ or } \left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ 0 & 1 & \frac{1}{4} & -2 \end{array} \right] \text{ or } \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 3 \\ 0 & 1 & \frac{1}{4} & -2 \end{array} \right]$$

(a) [2] What is the rank of A ?

[1] what is the rank of A ? 2 (+1) work

[1] How many variables describe? 3 (+1)

[1] Does # of free = # of variables - rank(A)? yes (+1)

9. Let the following vectors be in \mathbb{R}^2 .

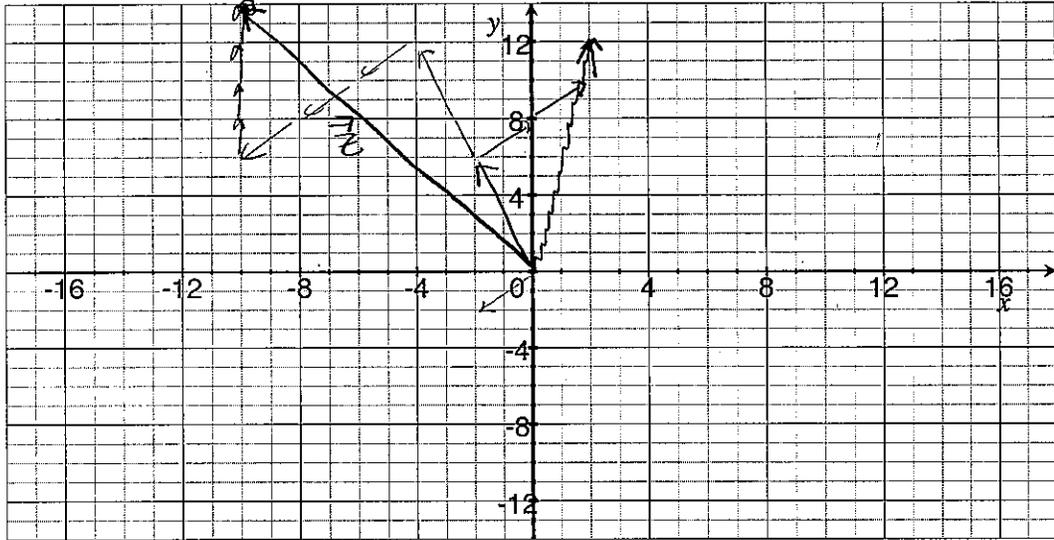
$$\vec{u} = [-2, 6]$$

$$\vec{v} = [2, 2]$$

$$\vec{w} = [0, 2]$$

(a) [4] Draw the vector $\vec{u} + 2\vec{v} + \vec{w}$ on the axes below.

$$\propto [2, 12]$$



[4] (b) ~~X~~ Is the angle between \vec{u} and $-\vec{v}$, obtuse, acute or right?

note: this can be done graphically

$$\vec{u} \cdot (-\vec{v}) = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \end{bmatrix} = 4 - 12 = -8$$

$$= \|\vec{u}\| \|\vec{v}\| \cos \theta$$

calc (4)

interpret (3)

$$\Rightarrow \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{-8}{\sqrt{4+36} \sqrt{4+4}} < 0 \Rightarrow \theta > 90^\circ \text{ obtuse}$$

(c) [5] Is $\vec{z} = \begin{bmatrix} -10 \\ 14 \end{bmatrix}$ a linear combination of \vec{u} , \vec{v} , and \vec{w} ? If so write \vec{z} as a linear combination of \vec{u} , \vec{v} , and \vec{w} . (Consider using work from the previous page.)

Note the work from the previous page directly applies (we were looking for $x_1, x_2 + x_3$ so that

$$x_1 \begin{bmatrix} -2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -10 \\ 14 \end{bmatrix}$$

so we've already found the solution set.

what looking for (2) found (3)

$$x_1 = 3 - \frac{1}{4}s$$

Let $s=4$

$$x_2 = -2 - \frac{1}{4}s$$

$$x_3 = s$$

then $x_1 = 2$
 $x_2 = -3$
and $x_3 = 4$

should work

Check:

$$2 \begin{bmatrix} -2 \\ 6 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 - 6 + 0 \\ 12 - 6 + 8 \end{bmatrix} = \begin{bmatrix} -10 \\ 14 \end{bmatrix}$$

note: there are many right answers

(d) [1] Verify your answer to (c) graphically.

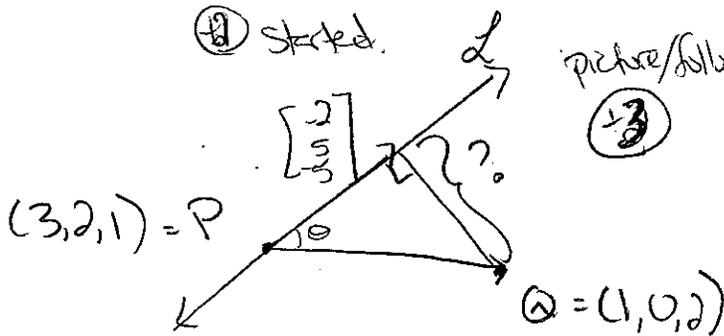
10. Let the $\vec{u} = [-2, 5, -5]$ be in \mathbb{R}^3 .

- (a) [2] Write the vector form for a line that is parallel to \vec{u} and passes through $(3, 2, 1)$.

$$(3, 2, 1) + t \begin{bmatrix} -2 \\ 5 \\ -5 \end{bmatrix}$$

- (b) [3] Let \mathcal{L} be the line that is described in (a). Find the distance from the point $Q = (1, 0, 2)$ to the line \mathcal{L} .

① started.



picture/diagram/plan

②

$$\vec{PQ} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

note

$$\cos \theta = \frac{\vec{PQ} \cdot \vec{u}}{\|\vec{PQ}\| \|\vec{u}\|}$$

$$= \frac{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ -5 \end{bmatrix}}{\sqrt{4+4+1} \sqrt{4+25+25}}$$

$$= \frac{4 - 10 - 5}{\sqrt{9} \sqrt{54}} = \frac{-11}{3\sqrt{54}}$$

and $\|\vec{PQ}\| = 3$

③ Calc formula

Sub into formula

$$\sin \theta = \frac{?}{\|\vec{PQ}\|} = \frac{?}{3}$$

$$? = 3 \cdot \sin \theta$$

$$= 3 \sqrt{1 - \cos^2 \theta}$$

$$= 3 \sqrt{1 - \left(\frac{-11}{3\sqrt{54}}\right)^2}$$

$$= 3 \sqrt{1 - \frac{121}{9 \cdot 54}}$$

$$= 3 \sqrt{\frac{486 - 121}{9 \cdot 54}} = 3 \frac{\sqrt{365}}{3\sqrt{54}}$$

$$= \frac{\sqrt{365}}{\sqrt{54}} \approx 2.59986$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-11}{3\sqrt{54}}\right)$$

best method ~~PN~~ ~~III~~ ~~III~~
new method ~~II~~

forget the sign

[9] 11. ~~Prove~~ Prove vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

started (+) Recall the def. of linear dependence: $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are lin dependent if $\exists c_1, \dots, c_k$, not all zero $\Rightarrow c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$, def of indep (+)

\Rightarrow If $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent, we need to show at least one of the vectors can be expressed as a linear combination of the others. right assumption & direction (+) logic (+)

directions (+) Since $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent, the definition implies $\exists c_1, \dots, c_k$ so that $c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$ (*) without loss of generality assume $c_1 \neq 0$. Then we can rearrange $*$ so that

$$-c_1 \vec{v}_1 = c_2 \vec{v}_2 + \dots + c_k \vec{v}_k \quad \text{alg (+)}$$

$$\Rightarrow \vec{v}_1 = -\frac{1}{c_1} c_2 \vec{v}_2 + \dots + -\frac{1}{c_1} c_k \vec{v}_k$$

Thus we wrote \vec{v}_1 as a linear combination of the other vectors.

\Leftarrow If one vector can be expressed as a linear combination of the others, we need to show the set is linearly dependent. right assumption & direction (+) logic (+) without loss of generality assume we can write v_1 as a linear combination of the others.

Then $\exists d_2, \dots, d_k$ so that

$$\vec{v}_1 = d_2 \vec{v}_2 + d_3 \vec{v}_3 + \dots + d_k \vec{v}_k, \quad \text{alg (+)}$$

Notice then if we subtract \vec{v}_1 from both sides we have: $\vec{0} = -\vec{v}_1 + d_2 \vec{v}_2 + d_3 \vec{v}_3 + \dots + d_k \vec{v}_k$

Thus $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ are linearly dependent. //