FINAL
Thur 6/11 @ 10:15am

Math 252

PRACTICE
57

NAME:

1. [10] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let a and b be constants with $a \leq b$ and $f(x)$ and $g(x)$ be continuous functions on $[a, b]$.

T F $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

T F $\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dx * g(x) + f(x) * \int_a^b g(x)\,dx$

T F $\frac{\alpha x^3+\beta x^2+\gamma}{x(x-1)^2(x^2+1)^3}$ can be put in the form of $\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$ for any constants α, β, and γ.

really

$\frac{A}{x} + \frac{B}{x-1} + \frac{E}{(x-1)^2} + \frac{Cx+D}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} + \frac{Hx+I}{(x^2+1)^3}$

T F If f is continuous, then $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx.$

T F If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both convergent, then $\int_a^\infty f(x) + g(x)\,dx$ is convergent.

$\int_a^\infty f(x)+g(x)dx = \lim_{t\to\infty}\int_a^t f(x)+g(x)dx = \lim_{t\to\infty}\left[\int_a^t f(x)dx + \int_a^t g(x)dx\right] \ldots$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [2] *Carefully* write down the first Fundamental Theorem of Calculus.

If f is continuous and $F(x) = \int_a^x f(t)dt$,

then F is con't and differentiable.

Furthermore $\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

3. [2] *Carefully* write down the second Fundamental Theorem of Calculus.

If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$\int_a^b f(x)dx = F(b)-F(c)$.

4. [5] $\frac{d}{dt}\int_0^t e^{x^2}\,dx$

e^{t^2} by FTC I

$\frac{d}{dx}\int_0^{x^2+3x} e^{t^2}\,dt$

Chain Rule

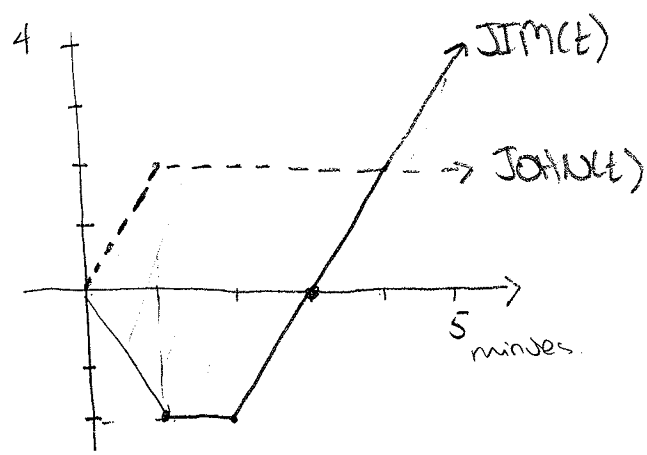
$[f(g(x))]' = f'(g(x)) \cdot g'(x).$

here $g(x) = x^2+3x$ $\quad g'(x) = 2x+3$

$f(x) = \int_0^x e^{t^2}\,dt$ $\quad f'(x) = e^{x^2}$ (by FTC I)

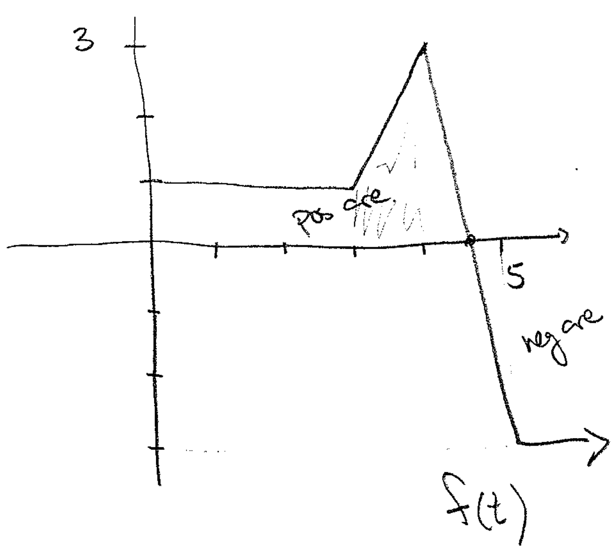
So $e^{g(x)^2} g'(x) = e^{(x^2+3x)^2} \cdot (2x+3)$

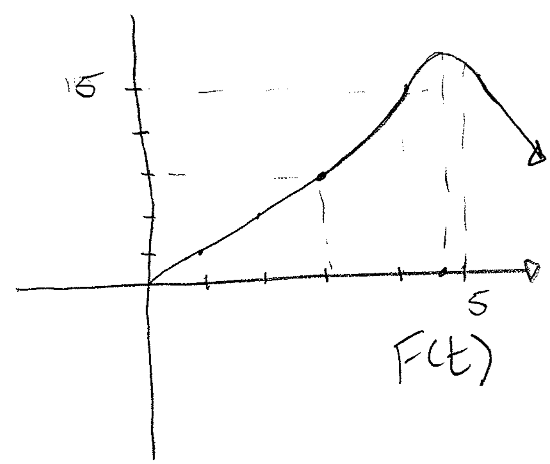
5. [5] The graphs of $JIM(t)$ and $JOHN(t)$ below trace the velocity of Jim and John respectively from time 0, measured in minutes. Explain what the physical meaning of $\int_0^5 JIM(t) - JOHN(t)\,dt$ is.



The signed area bounded between the graph of JIM and JOHN from 0 to 5 records the distance between Jim and John after 5 min assuming they've started in the same place at time 0.

6. [5] Let $F(x) = \int_0^x f(t)\,dt$ and $f(t)$ have the graph given below. Sketch the graph of $F(x)$.





7. Let $f(x) = \ln x$.

(a) [5] Find the average value of f on the interval $[1, e]$.

$$\frac{1}{e-1}\int_1^e \ln x \, dx = \frac{1}{e-1}\left[(\ln x)\cdot x\Big|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx\right] = \frac{1}{e-1}\left[x\ln x - x\right]_1^e$$

$$= \frac{1}{e-1}\left[(e\ln e - e) - (1\ln 1 - 1)\right]$$

$$= \frac{1}{e-1}\left[\cancel{(e-e)} - (\cancel{1\cdot 0} - 1)\right]$$

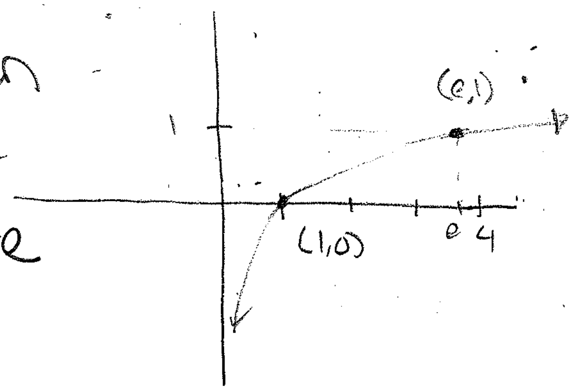
$$= \frac{1}{e-1}\left[1\right] = \frac{1}{e-1}$$

$u = \ln x$ $\quad v = x$

$du = \frac{1}{x}dx$ $\quad dv = dx$

(b) [5] Is there a number c between 1 and e so that $f(c)$ is equal to the value you found in part a? Explain, *briefly* why or why not.

Note $\ln x$ is con't on $[1, e]$.
We can thus use the mean value thm which states that there exists a c between 1 + e so that



$$f(c)\cdot[e-1] = \int_1^e \ln x \, dx$$

$$\Rightarrow f(c) = \frac{1}{e-1}\int_1^e \ln x \, dx = \frac{1}{e-1}.$$

Note: we didn't have to find it.

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

8. [10 *each*] Evaluate the following if they exist.

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan^4 x \, dx = \int_0^{\pi/4} \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \displaystyle\int_0^1 u^4(u^2+1)\, du = \int_0^1 u^6 + u^4 du$

$= \frac{1}{7}u^7 + \frac{1}{5}u^5 \Big]_0^1$

$= \frac{1}{7} + \frac{1}{5} = \frac{5+7}{35} = \frac{12}{35}$

(b) $\displaystyle\int x \cos^2 x \, dx = \int x \frac{1}{2}[1 + \cos 2x] dx = \frac{1}{2}\int x + x\cos 2x \, dx$

$= \frac{1}{2}\left[\int x\, dx + \int x \cos 2x \, dx\right] = \frac{1}{2}\left[\frac{1}{2}x^2 + x\frac{1}{2}\sin 2x - \int \frac{1}{2}\sin 2x \, dx\right]$

$u = x$ $\quad v = \frac{1}{2}\sin 2x$

$du = dx$ $\quad dv = \cos 2x \, dx$

$= \frac{1}{4}x^2 + x\frac{1}{4}\sin 2x - \frac{1}{4}\int \sin 2x \, dx$

$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{2}\cdot\frac{1}{2}\cos 2x + C$

(c) $\displaystyle\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx = \lim_{t\to\infty} \frac{-1}{x}\Big]_1^t$

$= \lim_{t\to\infty}\left(\frac{-1}{t} + \frac{+1}{1}\right) = \lim_{t\to\infty}\left[\frac{-1}{t} + 1\right]$

$= \lim_{t\to\infty}\frac{-1}{t} + \lim_{t\to\infty} 1$

$= 0 + 1 = 1$

$\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

$1+\cot^2\theta=\csc^2\theta$

(d) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta\,d\theta$

$u=2\tan\theta$

$du=2\sec^2\theta\,d\theta$

$\displaystyle= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{4\tan^2\theta\,\cancel{2}\,\cancel{\sec\theta}}\,d\theta$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$\displaystyle= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos^2\theta}}{\sin^2\theta}\,d\theta$

$\displaystyle= \frac{1}{4}\int \csc^2\theta\,d\theta = \boxed{-\frac{1}{4}\cot\theta+c}$

ck $\displaystyle\frac{d}{d\theta}(\cot\theta)=\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right)$

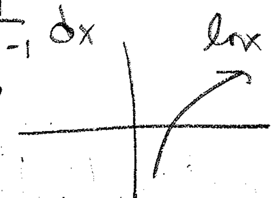
$\displaystyle=\frac{-\cos^2\theta}{\sin^2\theta}+\frac{-\sin\theta}{\sin\theta}=-(\cot^2\theta+1)$

$=-\csc^2\theta$ ✓

oops (e) $\cancel{\int_1^\infty \frac{1}{x^2}\,dx}$ howabout we look at $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ instead

note $\frac{1}{x-1}$ is not con't at $x=1$

$\displaystyle\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx + \lim_{s\to 1^+}\int_s^3 \frac{1}{x-1}\,dx$



$\displaystyle= \lim_{t\to 1^-} \ln|x-1|\Big]_0^t + \lim_{s\to 1^+} \ln|x-1|\Big]_s^3$

$\displaystyle= \underbrace{\lim_{t\to 1^-}\left[\ln|t-1|-\cancel{\ln 1}\right]}_{\text{diverges}} + \lim_{s\to 1^+}\left[\ln 2-\ln|s-1|\right]$ so $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ $\boxed{\text{diverges}}$

note $\displaystyle\lim_{t\to 1^-}\ln|t-1| = \ln\left(\lim_{t\to 1^-}|t-1|\right) \to -\infty$

(f) $\displaystyle\int \frac{17x-1}{2x^2+3x-2}\,dx$

$(2x-1)(x+2)$

$\displaystyle\frac{A}{2x-1}+\frac{B}{x+2}=\frac{17x-1}{(2x-1)(x+2)}$

$u=2x^2+3x-2$

$du=4x+3\,dx$

$\Rightarrow Ax+2A+B2x-B=17x-1$

$\Rightarrow \left.\begin{array}{l} A+2B=17 \\ 2A-B=-1 \end{array}\right\}$ $B=2A+1$

$A+2(2A+1)=17 \Rightarrow 5A+2=17$

$\Rightarrow 5A=15$

$A=3$

$\displaystyle\int \frac{17x-1}{(2x-1)(x+2)}\,dx = \int \frac{3}{2x-1}+\frac{7}{x+2}\,dx = 3\int\frac{1}{2x-1}\,dx + 7\int\frac{1}{x+2}\,dx$

$u=2x-1$

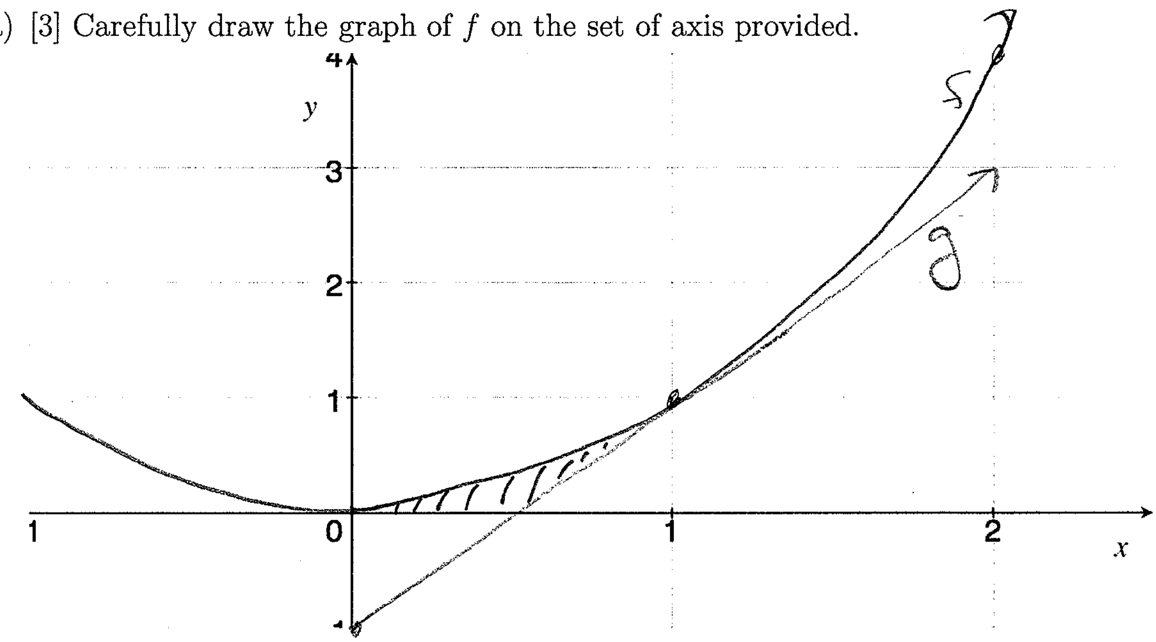
$du=2dx$

$\displaystyle= 3\int \frac{1}{u}\cdot\frac{1}{2}\,du + 7\ln|x+2|+c$

$\displaystyle\boxed{=\frac{3}{2}\ln|2x-1|+7\ln|x+2|+c}$

9. Let $f(x) = x^2$.

(a) [3] Carefully draw the graph of f on the set of axis provided.



(b) [4] Let g be the function tangent to f at $x = 1$. Find the rule for g and draw the graph of g on the above graph.

looking for m & b in $g(x) = mx + b$

$m = f'(1) = 2x\big|_{x=1} = 2\cdot 1 = 2$

$\Rightarrow g(x) = 2x + b$

g passes through the point $(1, f(1)) = (1,1)$ so

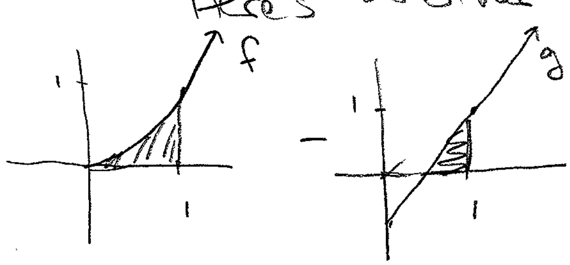
$1 = g(1) = 2(1) + b$

$\Rightarrow 1 = 2 + b \Rightarrow b = -1$

So $\boxed{g(x) = 2x - 1}$

(c) [6] Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

I wrote 2 different solutions to this on Midterm 2. Here's another solution one of you came up with:



$$= \int_0^1 x^2\,dx - \frac{1}{2}(1)\left(\frac{1}{2}\right)$$

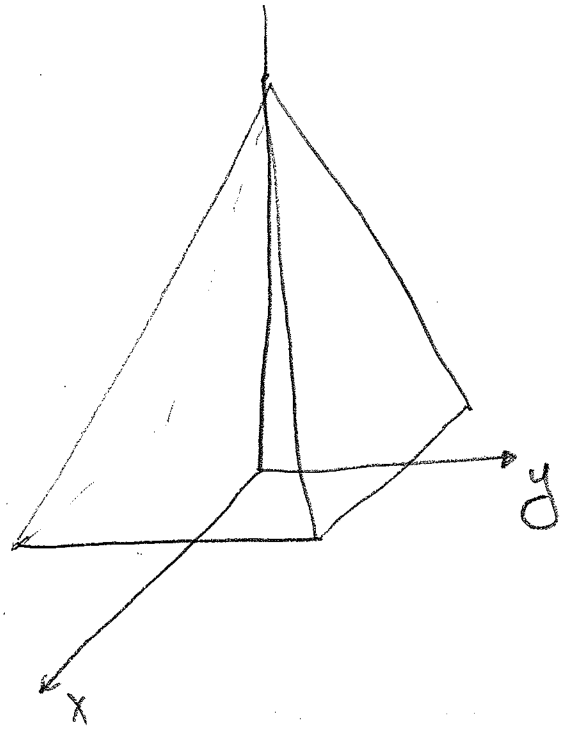
(height)(base)

$$= \frac{1}{3}x^3\Big]_0^1 - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}$$

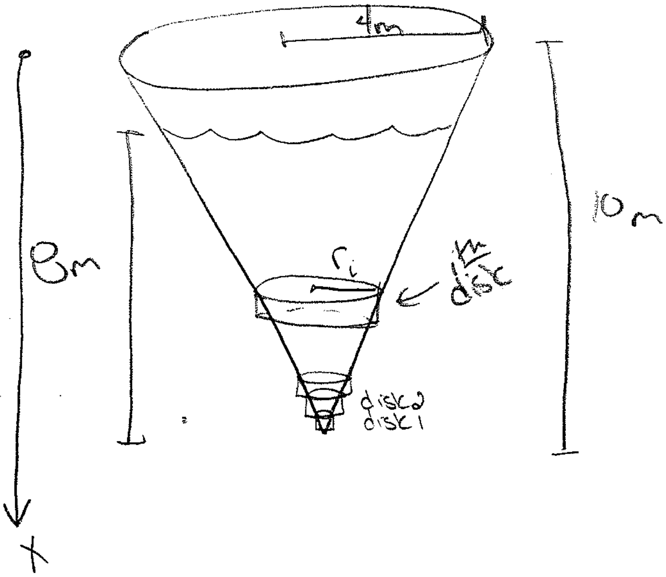
10. [10] Use calculus to show $\frac{L^2h}{3}$ is the volume of a pyramid whose base is a square with side L and whose height is h.

(solutions for this are on Midterm 2)

slice // to the x-y plane



11. [10] A tank has the shape of an inverted circular cone with height 10m and base 4 m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000kg/m^3.)



Work $\approx$ work to move disk 1 up + Work to move disk 2 up + ... + Work to move last disk up

Work $= \lim_{\#\text{ of disks}\to\infty}$ [Work to move disk 1 up + ... + work to move last disk up]

Work to move the i^{th} disk up = force to move i^{th} disk $\cdot$ dist

$= \left(\text{mass of } i^{th} \text{ disk}\right)\left(\text{acceleration}\right) \cdot x_i$

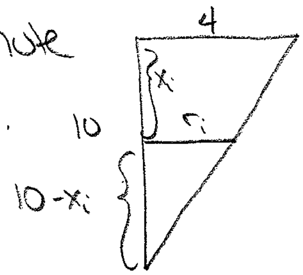
$= \left(\text{Volume of } i^{th} \text{ disk} \cdot 1000\right)(9.8) \cdot x_i$

$= \pi r_i^2 \cdot \Delta x \; 9800 \cdot x_i$

$= \pi \left(4 - \frac{2}{5} x_i\right)^2 \Delta x \cdot 9800 x_i$

need to write r_i as a function of x

note



by similar triangles $\frac{4}{r_i} = \frac{10}{10-x_i}$

$\Rightarrow \; 40 - 4x_i = r_i \cdot 10$

$\Rightarrow \; r_i = 4 - \frac{2}{5} x_i$

Returning to total work:

$$\int_2^{10} \pi\left(4-\frac{2}{5}x\right)^2 9800x \cdot dx = 9800\pi \int_2^{8} x\left(4-\frac{2}{5}x\right)^2 dx$$

$$= 9800\pi \int_2^{8} 16x - \frac{16}{5}x^2 + \frac{4}{25}x^5 \, dx = 9800\pi \left[8x^2 - \frac{16}{15}x^3 + \frac{4}{5^2 \cdot 6}x^6\right]_2^8 \; \ldots$$