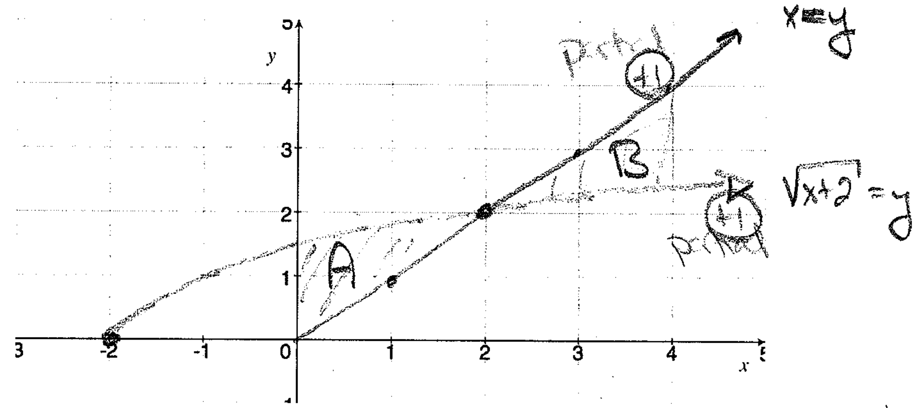
MIDTERM 2

Math 252

Spring 2009

Key

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.



Let A and B represent the positive area of the shaded regions each is in denoted areas (+1) partial

1.

(a) [3] Interpret $\int_0^4 \sqrt{x+2} - x\,dx$ as an area of a region.

A - B

involved area (+1)
signed area (+1)
signed area in the right way (+1)

if get partial $+\frac{1}{2}$ $+\frac{1}{2}$ $+\frac{1}{2}$

§6.1 #32

(b) [3] Interpret $\int_0^4 |\sqrt{x+2} - x|\,dx$ as an area of a region.

A + B

involved area (+1)
knew to make positive (+1)
got it (+1)

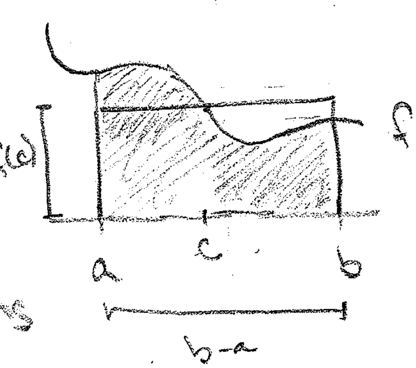
2. (a) [3] Write down the Mean Value Theorem for Integrals.

(+1) If f be a cont function on [a, b],
(+1) then there exists a # c between a + b (inclusive) so that
(+1) $\int_a^b f(x)\,dx = f(c)(b-a)$

§6.5 pg 443 + 444

(b) [2] What is the geometric interpretation?

there exists a value c
(+1) such that the shaded area is equal to
(+1) the area of a rectangle with dimensions (b-a) and f(c).



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§6.4 #8 with micro units

3. A reasonable rule to describe the force required to maintain a spring stretched x units beyond its natural length is given by Hooke's Law: $F(x) = kx$.

   (a) [2] A spring has a natural length of 20m (yes, it is a very large spring). If a 25-N force is required to keep it stretched to a length of 30 m, find the rule for the force function.

$+\frac{1}{2}$ $+1$

$25 = K(30-20)$

$25 = 10K$

$2.5 = K$

rule $F(x) = 2.5x$ $+\frac{1}{2}$

   (b) [4] How much work is required to stretch it from 20 m to 25m?

$$\text{Work} = \underbrace{\int_{20-20}^{25-20} 2.5x\,dx}_{+1 \quad +1} = \int_0^5 2.5x\,dx$$

integral/area $+\frac{1}{2}$ notation $+\frac{1}{2}$

$$= 2.5\,\frac{1}{2}\,x^2\Big]_0^5 = \frac{5}{2}\cdot\frac{1}{2}\left[25-0\right] = \frac{5}{4}\cdot 25$$

$+\frac{1}{2}$ $+\frac{1}{2}$

$$= \frac{125}{4} = 31.25\,J$$

partial $+\frac{1}{2}$ $W = Fd$

4. [4] Find the average value of the function $h(x) = \cos^4 x \sin x$ on the interval $[0, \pi]$.

§6.5 #7

$$\underbrace{\frac{1}{\pi-0}}_{+1}\underbrace{\int_0^{\pi}}_{+\frac{1}{2}}\underbrace{\cos^4 x\,\sin x\,dx}_{+\frac{1}{2}} = \underbrace{\frac{1}{\pi}\int_1^{-1}}_{+\frac{1}{2}}\underbrace{-u^4\,du}_{+\frac{1}{2}} = \frac{-1}{\pi}\,\frac{1}{5}\,u^5\Big]_1^{-1}$$

$+\frac{1}{2}$

working for sub $+\frac{1}{2}$

$u = \cos x$

$du = -\sin x\,dx$

$\Rightarrow -du = \sin x\,dx$

$$= \frac{-1}{5\pi}\left[(-1)^5 - (1)^5\right]$$

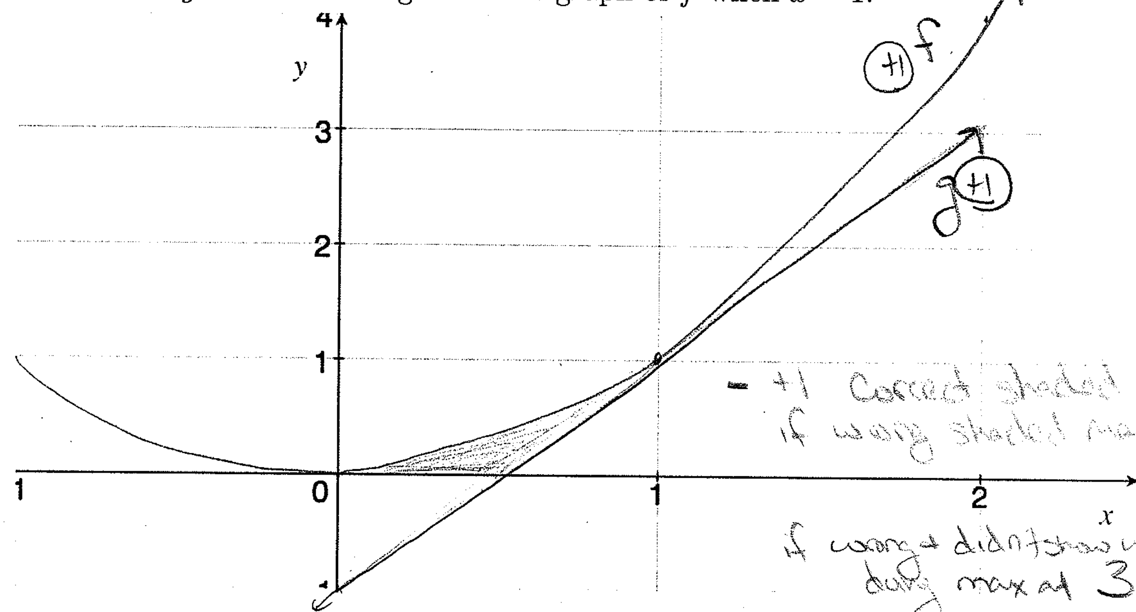
$$= \frac{2}{5\pi}$$

5. Let $f(x) = x^2$ and $g(x) = 2x - 1$.

§6.1 #48

(a) [3] Carefully draw the graph of f and g on the set of axis provided. It can be shown that g is the line tangent to the graph of f when $x = 1$.

partial - shapes (+1) ½ each



− +1 Correct shaded region if wrong shaded max at 4.5

if wrong + didn't show what doing max at 3

(b) [6] Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

with respect to y (+½)

$y = x^2 \Rightarrow \sqrt{y} = x$; $y = 2x - 1$, $\frac{y+1}{2} = x$

(+½) looking for x as a function of y

(+½) got them

approx rectangles (+½ picture)

$(\Delta y)\ell_1 + (\Delta y)\ell_2 + \dots (\Delta y)\ell_n$

ℓ_i is the dif in x coord of the 2 functions (+½)

$\Rightarrow \ell_i = \left(\frac{y+1}{2}\right) - \sqrt{y} = \frac{1}{2}y + \frac{1}{2} - y^{\frac{1}{2}}$

order (+½) difference (+½)

take a limit as $n \to \infty$

$\text{Area} = \int_0^1 \frac{1}{2}y + \frac{1}{2} - y^{\frac{1}{2}}\,dy = \frac{1}{2}\cdot\frac{1}{2}y^2 + \frac{1}{2}y - \frac{2}{3}y^{\frac{3}{2}}\Big]_0^1$ (+½)

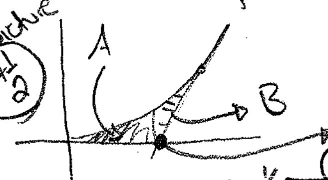
(+1) (+½)

$= \frac{1}{4} + \frac{1}{2} - \frac{2}{3} = \frac{3}{12} + \frac{6}{12} - \frac{8}{12} = \frac{1}{12}$

(+½)

with respect to x (+½)

(+½ picture)



$\text{Area} = A + B$ (+1)

$(\frac{1}{2}, 0)$ (+½)

$\text{Area } A = \int_0^{\frac{1}{2}} x^2\,dx = \frac{1}{3}x^3\Big]_0^{\frac{1}{2}}$ (+1) (+½)

$= \frac{1}{3}\left[\frac{1}{8} - 0\right] = \frac{1}{24}$

$\text{Area } B = \int_{\frac{1}{2}}^1 x^2 - (2x-1)\,dx = \int_{\frac{1}{2}}^1 x^2 - 2x + 1\,dx$ (+1) (+½)

$= \frac{1}{3}x^3 - x^2 + x\Big]_{\frac{1}{2}}^1 = \left(\frac{1}{3} - 1 + 1\right) - \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{2}\right)$

$= \frac{1}{3} - \frac{1}{24} + \frac{1}{4} - \frac{1}{2} = \frac{8 - 1 + 6 - 12}{24} = \frac{1}{24}$

$\text{Area} = A + B$ (+½)

$= \frac{1}{24} + \frac{1}{24} = \frac{2}{24} = \frac{1}{12}$

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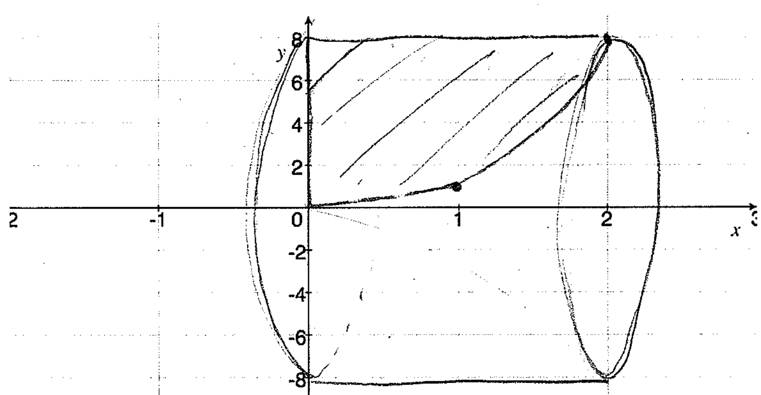
§6.3 #11

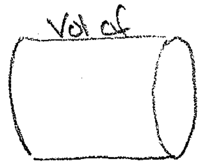
6. [7] Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the x-axis. Be clear about what methods you use (Are you using disks or cylindrical shells? Are you integrating with respect to x or y?)



If complete work will max at 5.5

Disk method (+½)

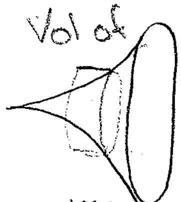
Vol of



[r −

h

Vol of



[r_i

Δx (+½)

pictures/notation (+1)

square right (+½)

formula (+½)

translated r (+1)

r_i is given by the y coord

approx disks:

$(\Delta x)\pi r_1^2 + (\Delta x)\pi r_2^2 + \dots + (\Delta x)\pi r_n^2$

$(\Delta x)\pi (x_1^3)^2 + (\Delta x)\pi (x_2^3)^2 + \dots$

$$\mathrm{Vol} = \underbrace{\pi r^2 \cdot h}_{(+1)} - \underbrace{\int_0^2}_{(+1)} \pi (x^3)^2\,dx$$

(+½)

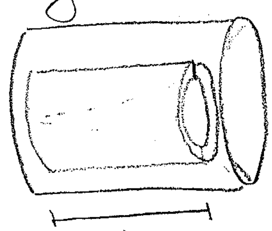
$$= \pi 8^2 \cdot 2 - \pi \int_0^2 x^6\,dx = 2\pi \cdot 64 - \pi \tfrac{1}{7} x^7 \Big]_0^2$$

$$= 2\pi \cdot 64 - \pi \tfrac{1}{7}\left[2^7 - 0^7\right] = 2\pi\left[64 - \tfrac{2^6}{7}\right]$$

(+½)

$$= 2\pi \cdot \frac{384}{7}$$

Cyc method (+½)



$[r_i [r_o$ $\exists \Delta y$

pictures/notation (+1)

h_i

approx cyc

$(h_1\pi r_o^2 - h_1\pi r_i^2) + (h_1\pi r_o^2 - h_2\pi r_i^2) + \dots$

examine the i^{th} cyl:

$h\pi(r_o^2 - r_i^2) = h_i\pi(r_o + r_i)(r_o - r_i)$ (+½)

formula (+½)

square right (+½)

$= h_i\pi(r_o + r_i)(\Delta y)$

(+1) h is the x coord of points in graph of $y = x^2$ so $h_i = \sqrt[3]{y_i}$

(+1) r_i's are moving up the y-axis so $r_i = y_i$

the i^{th} cyl: $\sqrt[3]{y_i}\,\pi 2 y_i (\Delta y)$

(+½)

$$\Rightarrow \underbrace{\int_0^8}_{(+1)} 2\pi y^{\frac{1}{3}} y\,dy = 2\pi \int_0^8 y^{\frac{4}{3}}\,dy = 2\pi \tfrac{3}{7} y^{\frac{7}{3}} \Big]_0^8$$

(+½)

$$= 2\pi \tfrac{3}{7}\left[8^{\frac{7}{3}} - 0\right] = 2\pi \cdot \tfrac{3}{7} 2^7$$

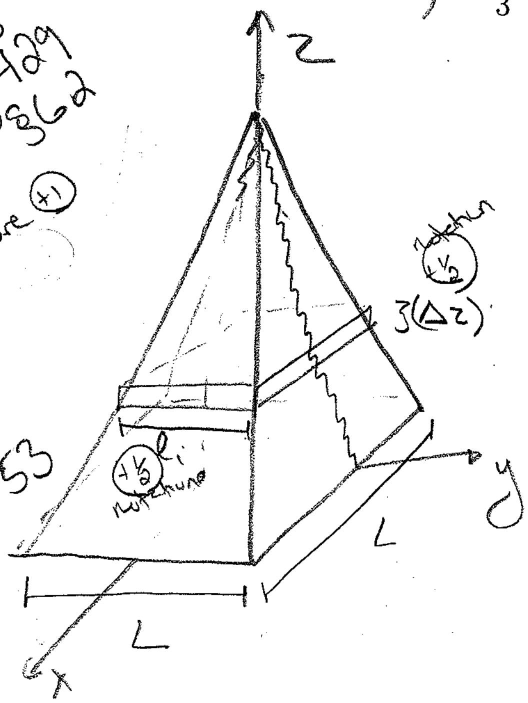
$$= 2\pi \frac{384}{7}$$

8 7. ~~[6]~~ Use calculus to show the volume of a pyramid, whose base is a square with side L and whose height is h, is $\dfrac{L^2 h}{3}$.

ex 8 429 pg 562

picture (+1)

very like §62 #53



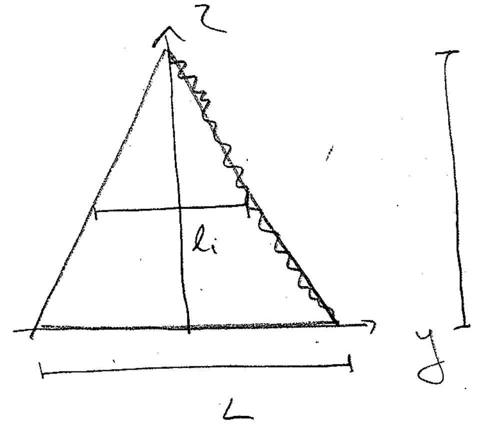
(+1) Slice // to xy plane

approx cyc (+1) formula involving squares

$(\Delta z)\cdot l_i \cdot l_i = (\Delta z) l_i^2$

(+1) need to get l as a function of z

↳ +½ if don't tell me but think they're...



note we can find the eq of

$z = my + b$

$\text{Slope} = m = \dfrac{-h}{\frac{1}{2}L} = -2\dfrac{h}{L}$

$\Rightarrow z = -2\dfrac{h}{L}y + b$

z intercept $= b = h$

$\Rightarrow z = \dfrac{-2h}{L}y + h$ (+1)

$\Rightarrow (z-h)\dfrac{L}{-2h} = y$

note l_i is twice the y coord (+½)

$\Rightarrow l_i = 2y = \cancel{2}(z-h)\dfrac{-L}{\cancel{2}h}$

$= \dfrac{-L}{h}z + L$

1.5

So approx cyc is:

$(\Delta z)(l_i)^2 = (\Delta z)\left(\dfrac{-L}{h}z + L\right)^2$

$\Rightarrow \text{Vol} = \underbrace{\int_0^h}_{(+1)} \left(\dfrac{-L}{h}z + L\right)^2 dz$

$= \displaystyle\int_0^h \dfrac{L^2}{h^2}z^2 - \dfrac{2L^2}{h}z + L^2\,dz$

$= \dfrac{L^2}{3h^2}z^3 + \dfrac{2L^2}{2h}z^2 + L^2 z\Big]_0^h$ (+½)

$= \dfrac{L^2}{3\cancel{h^2}}h^{\cancel{3}} - \dfrac{L^2}{\cancel{h}}h^{\cancel{2}} + L^2 h$

$= \dfrac{L^2 h}{3} - \cancel{L^2 h} + \cancel{L^2 h} = \dfrac{L^2 h}{3}$

8

8. Recall the Fundamental Theorem of Calculus Part 2: If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. Follow the steps below to show that *any* antiderivative will work.

fix

(a) [1] Let $g(x) = \int_a^x f(t)\,dt$. What is $g'(x)$?

$f(x)$ by FTC I

(b) [1] Let F be an antiderivative to f. What is $F'(x)$?

$f(x)$ by definition of an antiderivative

§5.3 pg 384

3 (c) ~~[4]~~ You know from Math 251 that if $F'(x) = g'(x)$ there exists a constant c so that $F(x) = g(x) + c$. Use this to show $F(b) - F(a) = \int_a^b f(x)\,dx$.

$$\begin{aligned} F(b)-F(a) &= [g(b)+c]-[g(a)+c] && (+1)\\ &= g(b)+\cancel{c}-g(a)-\cancel{c} && (+\tfrac{1}{2})\\ &= g(b)-g(a) && (+\tfrac{1}{2})\\ &= \int_a^b f(t)\,dt - \cancel{\int_a^a f(t)\,dt} && (+1)\\ &= \int_a^b f(t)\,dt. \end{aligned}$$

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