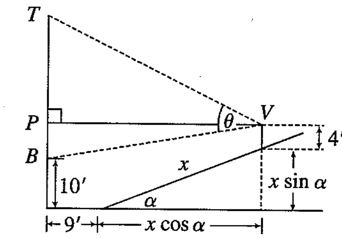
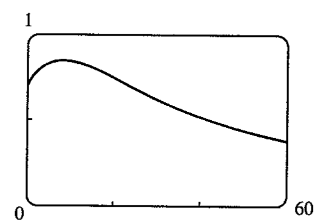
APPLIED PROJECT Where to Sit at the Movies

1.



$|VP| = 9 + x\cos\alpha$, $|PT| = 35 - (4 + x\sin\alpha) = 31 - x\sin\alpha$, and
$|PB| = (4 + x\sin\alpha) - 10 = x\sin\alpha - 6$. So using the Pythagorean Theorem, we have
$|VT| = \sqrt{|VP|^2 + |PT|^2} = \sqrt{(9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2} = a$, and
$|VB| = \sqrt{|VP|^2 + |PB|^2} = \sqrt{(9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2} = b$. Using the Law of Cosines on $\triangle VBT$, we
get $25^2 = a^2 + b^2 - 2ab\cos\theta \;\Leftrightarrow\; \cos\theta = \dfrac{a^2 + b^2 - 625}{2ab} \;\Leftrightarrow\; \theta = \arccos\left(\dfrac{a^2 + b^2 - 625}{2ab}\right)$, as required.

2. From the graph of θ, it appears that the value of x which maximizes θ is $x \approx 8.25$ ft. Assuming that the first row is at $x = 0$, the row closest to this value of x is the fourth row, at $x = 9$ ft, and from the graph, the viewing angle in this row seems to be about 0.85 radians, or about 49°.



3. With a CAS, we type in the definition of θ, substitute in the proper values of a and b in terms of x and $\alpha = 20° = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical root finder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 8.253062$, as approximated in Problem 2.

4. From the graph in Problem 2, it seems that the average value of the function on the interval $[0, 60]$ is about 0.6. We can use a CAS to approximate $\frac{1}{60}\int_0^{60} \theta(x)\,dx \approx 0.625 \approx 36°$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(60) \approx 0.38$ and, from Problem 2, the maximum value is about 0.85.

6 Review

CONCEPT CHECK

1. (a) See Section 6.1, Figure 2 and Equations 6.1.1 and 6.1.2.

(b) Instead of using "top minus bottom" and integrating from left to right, we use "right minus left" and integrate from bottom to top. See Figures 11 and 12 in Section 6.1.

2. The numerical value of the area represents the number of meters by which Sue is ahead of Kathy after 1 minute.

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Law of Cosines on $\triangle VBT$, we

$\left.\frac{a^2+b^2-625}{2ab}\right)$, as required.

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3. (a) See the discussion in Section 6.2, near Figures 2 and 3, ending in the Definition of Volume.

(b) See the discussion between Examples 5 and 6 in Section 6.2. If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi(\text{radius})^2$. If the cross-section is a washer, find the inner radius r_{in} and outer radius r_{out} and use $A = \pi\left(r_{\text{out}}^2\right) - \pi\left(r_{\text{in}}^2\right)$.

4. (a) $V = 2\pi rh\,\Delta r = $ (circumference) (height) (thickness)

(b) For a typical shell, find the circumference and height in terms of x or y and calculate $V = \int_a^b$ (circumference) (height) (dx or dy), where a and b are the limits on x or y.

(c) Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. $\int_0^6 f(x)\,dx$ represents the amount of work done. Its units are newton-meters, or joules.

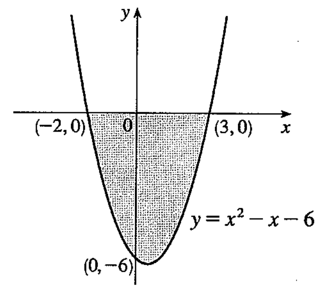
6. (a) The average value of a function f on an interval $[a, b]$ is $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

(b) The Mean Value Theorem for Integrals says that there is a number c at which the value of f is exactly equal to the average value of the function, that is, $f(c) = f_{\text{ave}}$. For a geometric interpretation of the Mean Value Theorem for Integrals, see Figure 2 in Section 6.5 and the discussion that accompanies it.

EXERCISES

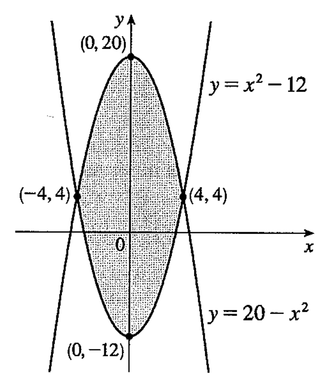
1. $0 = x^2 - x - 6 = (x-3)(x+2) \iff x = 3$ or -2. So

$$A = \int_{-2}^{3}\left[0 - \left(x^2 - x - 6\right)\right]dx = \int_{-2}^{3}\left(-x^2 + x + 6\right)dx$$
$$= \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + 6x\right]_{-2}^{3}$$
$$= \left(-9 + \tfrac{9}{2} + 18\right) - \left(\tfrac{8}{3} + 2 - 12\right)$$
$$= \tfrac{125}{6}$$



2. $20 - x^2 = x^2 - 12 \iff 32 = 2x^2 \iff x^2 = 16 \iff x = \pm 4$.
So

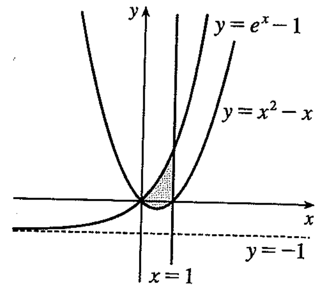
$$A = \int_{-4}^{4}\left[\left(20 - x^2\right) - \left(x^2 - 12\right)\right]dx = \int_{-4}^{4}\left(32 - 2x^2\right)dx$$
$$= 2\int_0^4\left(32 - 2x^2\right)dx \qquad \text{[even function]}$$
$$= 2\left[32x - \tfrac{2}{3}x^3\right]_0^4$$
$$= 2\left(128 - \tfrac{128}{3}\right) = \tfrac{512}{3}$$



3. $A = \int_0^1 \left[(e^x - 1) - (x^2 - x)\right] dx$

$= \int_0^1 (e^x - 1 - x^2 + x)\, dx = \left[e^x - x - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^1$

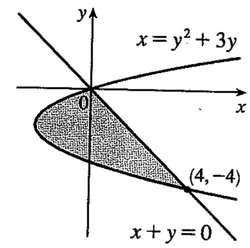
$= \left(e - 1 - \frac{1}{3} + \frac{1}{2}\right) - (1 - 0 - 0 + 0) = e - \frac{11}{6}$



4. $y^2 + 3y = -y \quad \Leftrightarrow \quad y^2 + 4y = 0 \quad \Leftrightarrow \quad y(y+4) = 0 \quad \Leftrightarrow$
$y = 0$ or -4.

$$A = \int_{-4}^{0} \left[-y - (y^2 + 3y)\right] dy = \int_{-4}^{0} (-y^2 - 4y)\, dy$$

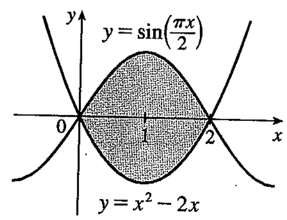
$= \left[-\frac{1}{3}y^3 - 2y^2\right]_{-4}^{0} = 0 - \left(\frac{64}{3} - 32\right) = \frac{32}{3}$



5. $A = \displaystyle\int_0^2 \left[\sin\left(\frac{\pi x}{2}\right) - (x^2 - 2x)\right] dx$

$= \left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}x^3 + x^2\right]_0^2$

$= \left(\frac{2}{\pi} - \frac{8}{3} + 4\right) - \left(-\frac{2}{\pi} - 0 + 0\right) = \frac{4}{3} + \frac{4}{\pi}$

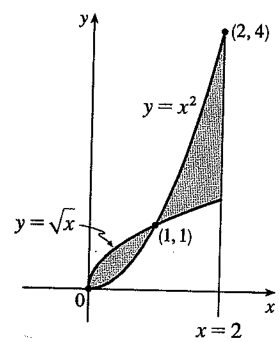


6. $A = \displaystyle\int_0^1 \left(\sqrt{x} - x^2\right) dx + \int_1^2 \left(x^2 - \sqrt{x}\right) dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 + \left[\frac{1}{3}x^3 - \frac{2}{3}x^{3/2}\right]_1^2$

$= \left[\left(\frac{2}{3} - \frac{1}{3}\right) - 0\right] + \left[\left(\frac{8}{3} - \frac{4}{3}\sqrt{2}\right) - \left(\frac{1}{3} - \frac{2}{3}\right)\right]$

$= \frac{10}{3} - \frac{4}{3}\sqrt{2}$

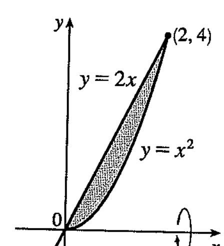


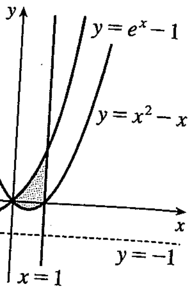
7. Using washers with inner radius x^2 and outer radius $2x$, we have

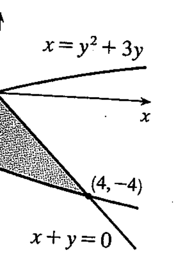
$$V = \pi \int_0^2 \left[(2x)^2 - \left(x^2\right)^2\right] dx = \pi \int_0^2 \left(4x^2 - x^4\right) dx$$

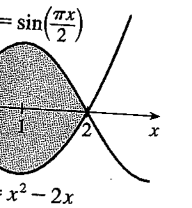
$= \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right) = 32\pi \cdot \frac{2}{15}$

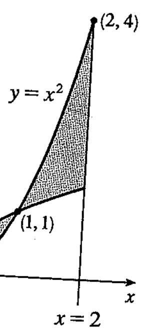
$= \frac{64\pi}{15}$

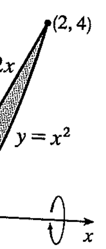






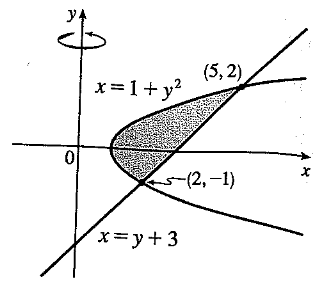




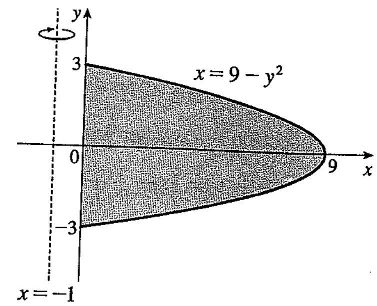


8. $1+y^2=y+3 \Leftrightarrow y^2-y-2=0 \Leftrightarrow (y-2)(y+1)=0 \Leftrightarrow y=2$ or -1.

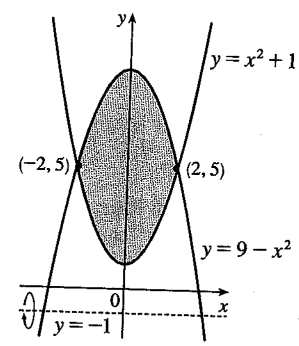
$$V=\pi\int_{-1}^{2}\left[(y+3)^2-\left(1+y^2\right)^2\right]dy$$
$$=\pi\int_{-1}^{2}(y^2+6y+9-1-2y^2-y^4)\,dy$$
$$=\pi\int_{-1}^{2}(8+6y-y^2-y^4)\,dy=\pi\left[8y+3y^2-\tfrac{1}{3}y^3-\tfrac{1}{5}y^5\right]_{-1}^{2}$$
$$=\pi\left[\left(16+12-\tfrac{8}{3}-\tfrac{32}{5}\right)-\left(-8+3+\tfrac{1}{3}+\tfrac{1}{5}\right)\right]$$
$$=\pi\left(33-\tfrac{9}{3}-\tfrac{33}{5}\right)=\tfrac{117\pi}{5}$$



9. $$V=\pi\int_{-3}^{3}\left\{\left[(9-y^2)-(-1)\right]^2-\left[0-(-1)\right]^2\right\}dy$$
$$=2\pi\int_{0}^{3}\left[(10-y^2)^2-1\right]dy$$
$$=2\pi\int_{0}^{3}(100-20y^2+y^4-1)\,dy$$
$$=2\pi\int_{0}^{3}(99-20y^2+y^4)\,dy=2\pi\left[99y-\tfrac{20}{3}y^3+\tfrac{1}{5}y^5\right]_{0}^{3}$$
$$=2\pi\left(297-180+\tfrac{243}{5}\right)=\tfrac{1656\pi}{5}$$



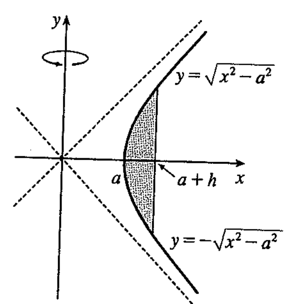
10. $$V=\pi\int_{-2}^{2}\left\{\left[(9-x^2)-(-1)\right]^2-\left[(x^2+1)-(-1)\right]^2\right\}dx$$
$$=\pi\int_{-2}^{2}\left[(10-x^2)^2-(x^2+2)^2\right]dx$$
$$=2\pi\int_{0}^{2}(96-24x^2)\,dx=48\pi\int_{0}^{2}(4-x^2)\,dx$$
$$=48\pi\left[4x-\tfrac{1}{3}x^3\right]_{0}^{2}=48\pi\left(8-\tfrac{8}{3}\right)=256\pi$$



11. The graph of $x^2-y^2=a^2$ is a hyperbola with right and left branches. Solving for y gives us $y^2=x^2-a^2 \Rightarrow y=\pm\sqrt{x^2-a^2}$. We'll use shells and the height of each shell is $\sqrt{x^2-a^2}-\left(-\sqrt{x^2-a^2}\right)=2\sqrt{x^2-a^2}$.

The volume is $V=\int_a^{a+h}2\pi x\cdot 2\sqrt{x^2-a^2}\,dx$. To evaluate, let $u=x^2-a^2$, so $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$. When $x=a$, $u=0$, and when $x=a+h$, $u=(a+h)^2-a^2=a^2+2ah+h^2-a^2=2ah+h^2$.

Thus, $V=4\pi\int_0^{2ah+h^2}\sqrt{u}\left(\frac{1}{2}\,du\right)=2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{2ah+h^2}=\frac{4}{3}\pi(2ah+h^2)^{3/2}$.



12. $V = \int_{3\pi/2}^{5\pi/2} 2\pi x \cos x \, dx$ [by the method of cylindrical shells]

13. $V = \int_0^1 \pi\left[\left(1-x^3\right)^2 - \left(1-x^2\right)^2\right] dx$

14. $V = \int_0^2 2\pi\left(8-x^3\right)(2-x)\,dx$

15. (a) A cross-section is a washer with inner radius x^2 and outer radius x.

$$V = \int_0^1 \pi\left[(x)^2 - \left(x^2\right)^2\right] dx = \int_0^1 \pi\left(x^2 - x^4\right) dx = \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 = \pi\left[\tfrac{1}{3} - \tfrac{1}{5}\right] = \tfrac{2\pi}{15}$$

(b) A cross-section is a washer with inner radius y and outer radius $\sqrt{y}$.

$$V = \int_0^1 \pi\left[\left(\sqrt{y}\right)^2 - y^2\right] dy = \int_0^1 \pi\left(y - y^2\right) dy = \pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1 = \pi\left[\tfrac{1}{2} - \tfrac{1}{3}\right] = \tfrac{\pi}{6}$$

(c) A cross-section is a washer with inner radius $2-x$ and outer radius $2-x^2$.

$$V = \int_0^1 \pi\left[\left(2-x^2\right)^2 - (2-x)^2\right] dx = \int_0^1 \pi\left(x^4 - 5x^2 + 4x\right) dx = \pi\left[\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 2x^2\right]_0^1$$
$$= \pi\left[\tfrac{1}{5} - \tfrac{5}{3} + 2\right] = \tfrac{8\pi}{15}$$

16. (a) $A = \int_0^1 \left(2x - x^2 - x^3\right) dx = \left[x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1 = 1 - \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{5}{12}$

(b) A cross-section is a washer with inner radius x^3 and outer radius $2x - x^2$, so its area is $\pi\left(2x-x^2\right)^2 - \pi\left(x^3\right)^2$.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left[\left(2x-x^2\right)^2 - \left(x^3\right)^2\right] dx = \int_0^1 \pi\left(4x^2 - 4x^3 + x^4 - x^6\right) dx$$
$$= \pi\left[\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{4}{3} - 1 + \tfrac{1}{5} - \tfrac{1}{7}\right) = \tfrac{41\pi}{105}$$

(c) Using the method of cylindrical shells,

$$V = \int_0^1 2\pi x\left(2x - x^2 - x^3\right) dx = \int_0^1 2\pi\left(2x^2 - x^3 - x^4\right) dx = 2\pi\left[\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^1$$
$$= 2\pi\left(\tfrac{2}{3} - \tfrac{1}{4} - \tfrac{1}{5}\right) = \tfrac{13\pi}{30}.$$

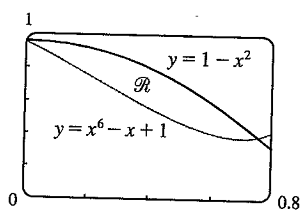
17. (a) Using the Midpoint Rule on $[0,1]$ with $f(x) = \tan\left(x^2\right)$ and $n = 4$, we estimate

$$A = \int_0^1 \tan\left(x^2\right) dx \approx \tfrac{1}{4}\left[\tan\left(\left(\tfrac{1}{8}\right)^2\right) + \tan\left(\left(\tfrac{3}{8}\right)^2\right) + \tan\left(\left(\tfrac{5}{8}\right)^2\right) + \tan\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{1}{4}(1.53) \approx 0.38$$

(b) Using the Midpoint Rule on $[0,1]$ with $f(x) = \pi\tan^2\left(x^2\right)$ (for disks) and $n = 4$, we estimate

$$V = \int_0^1 f(x)\,dx \approx \tfrac{1}{4}\pi\left[\tan^2\left(\left(\tfrac{1}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{3}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{5}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{\pi}{4}(1.114) \approx 0.87$$

18. (a)



From the graph, we see that the curves intersect at $x = 0$ and at $x = a \approx 0.75$, with $1 - x^2 > x^6 - x + 1$ on $(0, a)$.

(b) The area of $\mathcal{R}$ is

$$A = \int_0^a \left[\left(1-x^2\right) - \left(x^6 - x + 1\right)\right] dx = \left[-\tfrac{1}{3}x^3 - \tfrac{1}{7}x^7 + \tfrac{1}{2}x^2\right]_0^a \approx 0.12$$

(c) Using washers, the volume generated when $\mathcal{R}$ is rotated about the x-axis is

$$V = \pi\int_0^a \left[\left(1-x^2\right)^2 - \left(x^6 - x + 1\right)^2\right] dx = \pi\int_0^a \left(-x^{12} + 2x^7 - 2x^6 + x^4 - 3x^2 + 2x\right) dx$$
$$= \pi\left[-\tfrac{1}{13}x^{13} + \tfrac{1}{4}x^8 - \tfrac{2}{7}x^7 + \tfrac{1}{5}x^5 - x^3 + x^2\right]_0^a \approx 0.54$$

(d) Using shells, the volume generated when $\mathcal{R}$ is rotated about the y-axis is

$$V = \int_0^a 2\pi x\left[(1-x^2) - (x^6 - x + 1)\right] dx = 2\pi \int_0^a (-x^3 - x^7 + x^2)\, dx$$
$$= 2\pi\left[-\tfrac{1}{4}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{3}x^3\right]_0^a \approx 0.31$$

19. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the y-axis.

20. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \sqrt{2}\,\cos x\right\}$ about the x-axis.

21. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le y \le 2, 0 \le x \le 4 - y^2\right\}$ about the x-axis.

22. The solid is obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le x \le 1, 2 - \sqrt{x} \le y \le 2 - x^2\right\}$ about the x-axis.

Or: The solid is obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le x \le 1, x^2 \le y \le \sqrt{x}\right\}$ about the line $y = 2$.

23. Take the base to be the disk $x^2 + y^2 \le 9$. Then $V = \int_{-3}^{3} A(x)\, dx$, where $A(x_0)$ is the area of the isosceles right triangle whose hypotenuse lies along the line $x = x_0$ in the xy-plane. The length of the hypotenuse is $2\sqrt{9-x^2}$ and the length of each leg is $\sqrt{2}\sqrt{9-x^2}$. $A(x) = \frac{1}{2}\left(\sqrt{2}\sqrt{9-x^2}\right)^2 = 9 - x^2$, so

$$V = 2\int_0^3 A(x)\, dx = 2\int_0^3 (9 - x^2)\, dx = 2\left[9x - \tfrac{1}{3}x^3\right]_0^3 = 2(27-9) = 36.$$

24. $V = \int_{-1}^{1} A(x)\, dx = 2\int_0^1 A(x)\, dx = 2\int_0^1 \left[(2-x^2) - x^2\right]^2 dx = 2\int_0^1 \left[2(1-x^2)\right]^2 dx$
$= 8\int_0^1 (1 - 2x^2 + x^4)\, dx = 8\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = 8\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64}{15}$

25. Equilateral triangles with sides measuring $\frac{1}{4}x$ meters have height $\frac{1}{4}x \sin 60° = \frac{\sqrt{3}}{8}x$. Therefore,

$A(x) = \frac{1}{2}\cdot\frac{1}{4}x\cdot\frac{\sqrt{3}}{8}x = \frac{\sqrt{3}}{64}x^2$. $V = \int_0^{20} A(x)\, dx = \frac{\sqrt{3}}{64}\int_0^{20} x^2\, dx = \frac{\sqrt{3}}{64}\left[\frac{1}{3}x^3\right]_0^{20} = \frac{8000\sqrt{3}}{64\cdot 3} = \frac{125\sqrt{3}}{3}\ \text{m}^3$.

26. (a) By the symmetry of the problem, we consider only the solid to the right of the origin. The semicircular cross-sections perpendicular to the x-axis have radius $1 - x$, so $A(x) = \frac{1}{2}\pi(1-x)^2$. Now we can calculate

$$V = 2\int_0^1 A(x)\, dx = 2\int_0^1 \tfrac{1}{2}\pi(1-x)^2\, dx = \int_0^1 \pi(1-x)^2\, dx = -\tfrac{\pi}{3}\left[(1-x)^3\right]_0^1 = \tfrac{\pi}{3}.$$

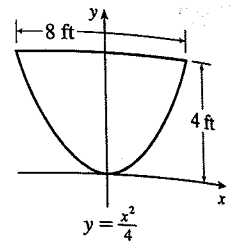
(b) Cut the solid with a plane perpendicular to the x-axis and passing through the y-axis. Fold the half of the solid in the region $x \le 0$ under the xy-plane so that the point $(-1, 0)$ comes around and touches the point $(1, 0)$. The resulting solid is a right circular cone of radius 1 with vertex at $(x, y, z) = (1, 0, 0)$ and with its base in the yz-plane, centered at the origin. The volume of this cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\cdot 1^2\cdot 1 = \frac{\pi}{3}$.

27. $f(x) = kx \quad\Rightarrow\quad 30\ \text{N} = k(15-12)\ \text{cm} \quad\Rightarrow\quad k = 10\ \text{N/cm} = 1000\ \text{N/m}.\ 20\ \text{cm} - 12\ \text{cm} = 0.08\ \text{m} \quad\Rightarrow$

$W = \int_0^{0.08} kx\, dx = 1000\int_0^{0.08} x\, dx = 500\left[x^2\right]_0^{0.08} = 500(0.08)^2 = 3.2\ \text{N-m} = 3.2\ \text{J}.$

28. The work needed to raise the elevator alone is 1600 lb × 30 ft = 48,000 ft-lb. The work needed to raise the bottom 170 ft of cable is 170 ft × 10 lb/ft × 30 ft = 51,000 ft-lb. The work needed to raise the top 30 ft of cable is $\int_0^{30} 10x\, dx = \left[5x^2\right]_0^{30} = 5\cdot 900 = 4500$ ft-lb. Adding these, we see that the total work needed is 48,000 + 51,000 + 4,500 = 103,500 ft-lb.

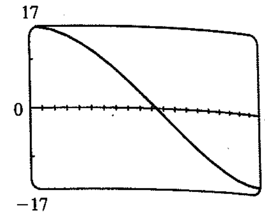
29. (a) The parabola has equation $y = ax^2$ with vertex at the origin and passing through $(4, 4)$. $4 = a \cdot 4^2 \Rightarrow a = \frac{1}{4} \Rightarrow y = \frac{1}{4}x^2 \Rightarrow x^2 = 4y \Rightarrow x = 2\sqrt{y}$. Each circular disk has radius $2\sqrt{y}$ and is moved $4 - y$ ft.



$$W = \int_0^4 \pi \left(2\sqrt{y}\right)^2 62.5\,(4-y)\,dy = 250\pi \int_0^4 y(4-y)\,dy$$

$$= 250\pi\left[2y^2 - \tfrac{1}{3}y^3\right]_0^4 = 250\pi\left(32 - \tfrac{64}{3}\right) = \tfrac{8000\pi}{3} \approx 8378 \text{ ft-lb}$$

(b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level — call it h) unknown: $W = 4000 \Leftrightarrow 250\pi\left[2y^2 - \frac{1}{3}y^3\right]_h^4 = 4000 \Leftrightarrow$ $\frac{16}{\pi} = \left[\left(32 - \frac{64}{3}\right) - \left(2h^2 - \frac{1}{3}h^3\right)\right] \Leftrightarrow h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0$.



We graph the function $f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi}$ on the interval $[0, 4]$ to see where it is 0. From the graph, $f(h) = 0$ for $h \approx 2.1$. So the depth of water remaining is about 2.1 ft.

30. $f_{\text{ave}} = \frac{1}{10-0}\int_0^{10} t\sin(t^2)\,dt = \frac{1}{10}\int_0^{100}\sin u\left(\frac{1}{2}\,du\right) \qquad [u = t^2,\ du = 2t\,dt]$

$= \frac{1}{20}\left[-\cos u\right]_0^{100} = \frac{1}{20}(-\cos 100 + \cos 0) = \frac{1}{20}(1 - \cos 100) \approx 0.007$

31. $\lim\limits_{h\to 0} f_{\text{ave}} = \lim\limits_{h\to 0} \dfrac{1}{(x+h)-x}\displaystyle\int_x^{x+h} f(t)\,dt = \lim\limits_{h\to 0}\dfrac{F(x+h)-F(x)}{h}$, where $F(x) = \int_a^x f(t)\,dt$. But we recognize this limit as being $F'(x)$ by the definition of a derivative. Therefore, $\lim\limits_{h\to 0} f_{\text{ave}} = F'(x) = f(x)$ by FTC1.

32. (a) $\mathfrak{R}_1$ is the region below the graph of $y = x^2$ and above the x-axis between $x = 0$ and $x = b$, and $\mathfrak{R}_2$ is the region to the left of the graph of $x = \sqrt{y}$ and to the right of the y-axis between $y = 0$ and $y = b^2$. So the area of $\mathfrak{R}_1$ is $A_1 = \int_0^b x^2\,dx = \left[\frac{1}{3}x^3\right]_0^b = \frac{1}{3}b^3$, and the area of $\mathfrak{R}_2$ is $A_2 = \int_0^{b^2}\sqrt{y}\,dy = \left[\frac{2}{3}y^{3/2}\right]_0^{b^2} = \frac{2}{3}b^3$. So there is no solution to $A_1 = A_2$ for $b \neq 0$.

(b) Using disks, we calculate the volume of rotation of $\mathfrak{R}_1$ about the x-axis to be $V_{1,x} = \pi\int_0^b \left(x^2\right)^2 dx = \frac{1}{5}\pi b^5$. Using cylindrical shells, we calculate the volume of rotation of $\mathfrak{R}_1$ about the y-axis to be $V_{1,y} = 2\pi\int_0^b x\left(x^2\right)dx = 2\pi\left[\frac{1}{4}x^4\right]_0^b = \frac{1}{2}\pi b^4$. So $V_{1,x} = V_{1,y} \Leftrightarrow \frac{1}{5}\pi b^5 = \frac{1}{2}\pi b^4 \Leftrightarrow 2b = 5 \Leftrightarrow b = \frac{5}{2}$. So the volumes of rotation about the x- and y-axes are the same for $b = \frac{5}{2}$.

(c) We use cylindrical shells to calculate the volume of rotation of $\mathfrak{R}_2$ about the x-axis: $\mathfrak{R}_{2,x} = 2\pi\int_0^{b^2} y\left(\sqrt{y}\right)dy = 2\pi\left[\frac{2}{5}y^{5/2}\right]_0^{b^2} = \frac{4}{5}\pi b^5$. We already know the volume of rotation of $\mathfrak{R}_1$ about the x-axis from part (b), and $\mathfrak{R}_{1,x} = \mathfrak{R}_{2,x} \Leftrightarrow \frac{1}{5}\pi b^5 = \frac{4}{5}\pi b^5$, which has no solution for $b \neq 0$.

(d) We use disks to calculate the volume of rotation of $\mathfrak{R}_2$ about the y-axis: $\mathfrak{R}_{2,y} = \pi\int_0^{b^2}\left(\sqrt{y}\right)^2 dy = \pi\left[\frac{1}{2}y^2\right]_0^{b^2} = \frac{1}{2}\pi b^4$. We know the volume of rotation of $\mathfrak{R}_1$ about the y-axis from part (b), and $\mathfrak{R}_{1,y} = \mathfrak{R}_{2,y} \Leftrightarrow \frac{1}{2}\pi b^4 = \frac{1}{2}\pi b^4$. But this equation is true for all b, so the volumes of rotation about the y-axis are equal for all values of b.