MIDTERM 2 **Math 251** Practice

Key

Note: This is a practice midterm and is intended only for study purposes. The actual exam will contain different questions and perhaps a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be functions.

T F $\frac{d}{dx}b^c = cb^{c-1}$ for a fixed b and c — d/dx of a constant is 0

T F $(x+y)^2 = x^2 + y^2$ — $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

T F $\frac{d}{dx}2^x = x2^{x-1}$ — $d/dx(2^x) = 2^x \ln 2$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. Find the following:

$$\lim_{x\to 0} \frac{3\sin(4x)}{2\sin(3x)}$$

$$= \lim_{x\to 0} \frac{3\sin(4x)}{2} \cdot \frac{1}{\sin 3x} \frac{\left(\frac{1}{3x}\right)}{\left(\frac{1}{3x}\right)}$$

$$= \lim_{x\to 0} \frac{3\sin 4x}{2\cdot 3x} \cdot \frac{1}{\left(\frac{\sin 3x}{3x}\right)}$$

$$= \lim_{x\to 0} \frac{3\sin 4x}{6x} \cdot \lim_{x\to 0} \frac{1}{\cancel{\left(\frac{\sin 3x}{3x}\right)}} \to 1$$

$$= \lim_{x\to 0} \frac{\sin 4x}{2x} \frac{(2)}{(2)}$$

$$= \lim_{x\to 0} 2\left(\frac{\sin 4x}{4x}\right)$$

$$= \lim_{x\to 0} 2 \cdot \lim_{x\to 0} \cancel{\frac{\sin 4x}{4x}} \to 1$$

$$= 2$$

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} \frac{(\cos x + 1)}{(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{\sin x(\cos x + 1)}$$

recall $\sin^2 x + \cos^2 x = 1$

$\Rightarrow -\sin^2 x = \cos^2 x - 1$

so $\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{-\sin^2 x}{\sin x(\cos x + 1)}$

$$= \lim_{x\to 0} \frac{-\sin x}{\cos x + 1}$$

$$= \frac{\lim_{x\to 0} -\sin x}{\lim_{x\to 0}(\cos x + 1)} = \frac{0}{2}$$

$$= 0$$

3. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$ where h is:

$h(x) = 5f(x) - 4g(x)$

$$h'(x) = \tfrac{d}{dx}(5f(x) - 4g(x))$$
$$= 5\tfrac{d}{dx}f(x) - 4\tfrac{d}{dx}(g(x))$$
$$= 5f'(x) - 4g'(x)$$
$$h'(2) = 5f'(2) - 4g'(2)$$
$$= 5(-2) - 4(7) = -10 - 28 = 38$$

$h(x) = f(x)g(x)$

$$h'(x) = f(x)\cdot g'(x) + f'(x)g(x)$$
$$h'(2) = f(2)g'(2) + f'(2)g(2)$$
$$= (-3)(7) + (-2)(4)$$
$$= -21 - 8 = -29$$

$h(x) = \frac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$
$$h'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2}$$
$$= \frac{4\cdot(-2) - (-3)(7)}{4^2} = \frac{-8+21}{16} = \frac{13}{16}$$

$h(x) = \frac{g(x)}{1+f(x)}$

$$h'(x) = \frac{[1+f(x)]g'(x) - g(x)\tfrac{d}{dx}(1+f(x))}{[1+f(x)]^2}$$
$$= \frac{[1+f(x)]g'(x) - g(x)\cdot f'(x)}{[1+f(x)]^2}$$
$$h'(2) = \frac{(1+-3)(7) - 4(-2)}{(1+(-3))^2} = \frac{-14+8}{4} = \frac{-6}{4}$$
$$= -3/2$$

4. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

$$F'(x) = f'(g(x))\cdot g'(x)$$
$$F'(5) = f'(g(5))g'(5) = f'(-2)\cdot 6 = 4\cdot 6 = 24$$

5. If $G(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $G'(1)$.

$$\tfrac{d}{dx}(G(x)) = f'(g(x))\cdot g'(x) = f'(x\cdot f(xf(x)))g'(x) = f'(x\cdot f(xf(x)))\,g'(x)$$

$f(x) = f(x)$ $\quad f'(x) = f'(x)$

$g(x) = x\cdot f(xf(x))$ $\quad g'(x) = ? = f(xf(x)) + xf'(xf(x))(xf'(x) + f(x))$

$$G'(1) = f'(f(f(1)))g'(1)$$
$$= f'(f(2))g'(1)$$
$$= f'(3)[f(f(1)) + f'(f(1))(f'(1) + f(1))]$$
$$= 6[3 + 5(4+2)]$$
$$= 18 + 30(6) = 198$$

$$? = \tfrac{d}{dx}(x\cdot f(xf(x)))$$
$$= \tfrac{d}{dx}(x)\cdot f(xf(x)) + x\cdot\tfrac{d}{dx}(f(xf(x)))$$
$$= f(xf(x)) + x\underbrace{\tfrac{d}{dx}(f(xf(x)))}_{*}$$
$$= f(xf(x)) + x f'(x\cdot f(x))(xf'(x) + f(x)) \Leftarrow$$

$$* = \tfrac{d}{dx}(f(xf(x)))$$

$\mathcal{F}(x) = f(x)$ $\quad \mathcal{F}'(x) = f'(x)$

$\mathcal{G}(x) = x\cdot f(x)$ $\quad \mathcal{G}'(x) = x\cdot\tfrac{d}{dx}f(x) + \tfrac{d}{dx}(x)\cdot f(x)$

$$= x\cdot f'(x) + f(x)$$

$$* = \mathcal{F}'(\mathcal{G}(x))\mathcal{G}'(x) = \mathcal{F}'(xf(x))\cdot(xf'(x) + f(x))$$
$$= f(x\cdot f(x))\cdot(xf'(x) + f(x))$$

6. Find the $\frac{dy}{dx}$ of the following:

$$y = \frac{\sin(x) + x^2\cos(x)}{\cos(\frac{1}{x})}$$

$= [\sin(x) + x^2\cos(x)][\cos\frac{1}{x}]^{-1}$

$\frac{dy}{dx} = [\sin(x) + x^2\cos x]\frac{d}{dx}(\cos\frac{1}{x})^{-1} + \frac{d}{dx}[\sin(x) + x^2\cos x](\cos\frac{1}{x})^{-1}$

$= [\sin x + x^2\cos x] \cdot (\cos\frac{1}{x})^{-2}\sin\frac{1}{x} \cdot -x^{-2}$

$+ (\cos x + (-x^2\sin x + 2x\cos x))(\cos\frac{1}{x})^{-1}$

$\frac{dy}{dx} = -\frac{(\sin x + x^2\cos x)\sin\frac{1}{x}}{(\cos^2\frac{1}{x}) \cdot x^2} + \frac{\cos x - x^2\sin x + 2x\cos x}{\cos\frac{1}{x}}$

$$y = (2x^2 + \ln(7x^2))(e^x - 4)$$

ln

$\ln y = \ln[2x^2 + \ln(7x^2)] + \ln(e^x - 4)$

$\frac{d}{dx}$

$\frac{1}{y}\frac{dy}{dx} = \frac{4x + \frac{1}{7x^2} \cdot 14x}{2x^2 + \ln(7x^2)} + \frac{1}{e^x - 4} \cdot e^x$

$\frac{dy}{dx} = y\left(\frac{4x + \frac{2}{x}}{2x^2 + \ln(7x^2)} + \frac{e^x}{e^x - 4}\right)$

$$y = \frac{x^{\frac{1}{4}}\sqrt{x^4 + 2}}{(4x - 3)^7}$$

ln

$\ln y = \ln x^{1/4} + \ln(x^4 + 2)^{1/2} - \ln(4x - 3)^7$

$\ln y = \frac{1}{4}\ln x + \frac{1}{2}\ln(x^4 + 2) - 7\ln(4x + 3)$

$\frac{d}{dx}$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4} \cdot \frac{1}{x} + \frac{1}{2}\frac{1}{x^4 + 2} \cdot 4x^3 - 7\frac{1}{4x + 3} \cdot 4$

$\frac{dy}{dx} = y\left(\frac{1}{4x} + \frac{4x^3}{2(x^4 + 2)} - \frac{28}{4x + 3}\right)$

$= \frac{x^{1/4}\sqrt{x^4 + 2}}{(4x - 3)^7}\left(\frac{1}{4x} + \frac{4x^3}{2(x^4 + 2)} - \frac{28}{4x + 3}\right)$

either would do given the instructions.

$$x^2 + y^2 = 25$$

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$= \frac{-x}{y}$

$$y = \sin(e^{\ln(x^2)}) = \sin(x^2)$$

$f(x) = \sin x$ $\quad f'(x) = \cos x$

$g(x) = x^2$ $\quad g'(x) = 2x$

$\frac{dy}{dx} = f'(g(x))\, g'(x)$

$= f'(x^2) \cdot 2x$

$= (\cos x^2) \cdot 2x$

$= 2x\cos x^2$

$$y = (\sin x)^{\ln x}$$

ln

$\ln y = \ln(\sin x)^{\ln x}$

$= (\ln x)\ln(\sin x)$

$\frac{d}{dx}$

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\ln x) \cdot \ln(\sin x) + \ln x\frac{d}{dx}(\ln \sin x)$

$= \frac{1}{x} \cdot \ln\sin x + (\ln x)\frac{1}{\sin x} \cdot \cos x$

$\Rightarrow \frac{dy}{dx} = y\left(\frac{\ln\sin x}{x} + (\ln x)\cot x\right)$

$= (\sin x)^{\ln x}\left(\frac{\ln\sin x}{x} + (\cot x)(\ln x)\right)$

7. Find the equations of all lines tangent to the curve described by the relation $x^2y^2+xy = 2$ that ~~are also parallel to the~~ line described by $y = -x - \pi$.

$$\frac{d}{dx}(x^2y^2+xy) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(x^2)\cdot y^2 + x^2\cdot\frac{d}{dx}(y^2) + \frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y) = 0$$

$$2x\cdot y^2 + \underline{x^2\cdot 2y\frac{dy}{dx}} + y + \underline{x\frac{dy}{dx}} = 0$$

$$\frac{dy}{dx}(2x^2y+x) = -2xy^2-y$$

$$\frac{dy}{dx} = \frac{-2xy^2-y}{2x^2y+x}$$

want to find pts when $dy/dx = -1$

$$-1 = \frac{-2xy^2-y}{2x^2y+x}$$

$$2x^2y+x = 2xy^2+y$$

$$x(2xy+1) = y(2xy+1)$$

$$x(2xy+1) - y(2xy+1) = 0$$

$$(x-y)(2xy+1) = 0$$

$\Rightarrow x=y$ or $2xy=-1$

$xy = -\frac{1}{2}$

if $xy = -\frac{1}{2}$

then

$(xy)^2+xy = \frac{1}{4} - \frac{1}{2} \neq 2$

so we must have

$x=y$

to find these points we need

$(xy)^2+xy = 2$

$\Rightarrow x^4+x^2 = 2$

$x^4+x^2-2=0$

$(x^2+2)(x^2-1)=0$

$\Rightarrow x^2+2=0$ or $x^2-1=0$

$x^2=-2$ $\qquad x^2=1$

$\cancel{x=\pm\sqrt{-2}}$ $\qquad x=\pm 1$

So happens at 2 points

$(1,1)$ & $(-1,-1)$

$y = m_1x+b_1$

$1 = -1(1)+b_1$

$1 = -1+b_1$

$2 = b_1$

So

$y = -x+2$

$y = m_2x+b_2$

$-1 = (-1)(-1)+b_2$

$-1 = 1+b_2$

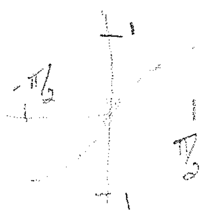
$-2 = b_2$

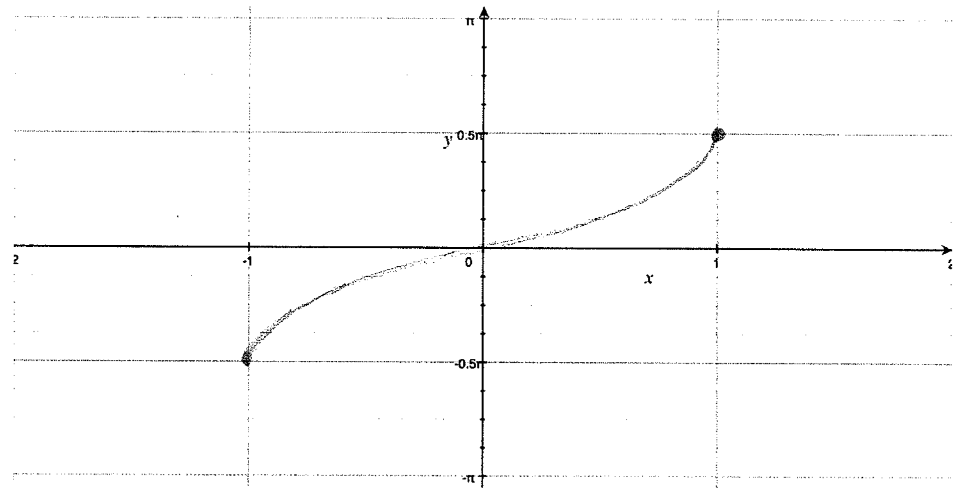
So

$y = -x-2$

8. Consider the relation $y = \arcsin x$. The following problem will step you through the *proof* that $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$.

(a) Draw the graph of $y = \arcsin x$ in the space provided below.





(b) What is the domain of arcsin? Ther range is between $-\pi/2$ & $\pi/2$

$[-1, 1]$

(c) Use Implicit Differentiation to find $\frac{dy}{dx}$ in terms of x and y.

$y = \arcsin x$

$\Leftrightarrow \quad \sin y = x \quad \downarrow d/dx$

$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$

$f(x) = \sin x \quad f'(x) = \cos x$

$g(x) = y \quad g'(x) = dy/dx$

$\frac{d}{dx}(\sin y) = f'(y) \cdot dy/dx = (\cos y)\frac{dy}{dx}$

so

$dy/dx \cdot \cos y = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

(d) Use simplification procedures and trig identities to wrtie $\frac{dy}{dx}$ in terms of only x. Cite when the domain restriction of arcsin was necessary.

recall $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \cos^2\theta = 1 - \sin^2\theta$

Since y is between $-\pi/2$ & $\pi/2$, $\cos y \geq 0$

$\Rightarrow \cos y = \sqrt{1 - \sin^2 y}$

thus

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$