FINAL

Math 251

Practice

Note: This is a practice final and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be differentiable functions and h be a constant.

   T F $\frac{x+h}{2x} = \frac{1+h}{x}$

   T F $\sqrt{x^2+h^2} = x+h$

   T F $\lim_{x\to r} f(x) = f(r)$ for all r in the domain of f.

   f is diff $\Rightarrow$ f is con't

   T F If $\lim_{x\to r} g(x) = 0$, then $\lim_{x\to r} \frac{f(x)}{g(x)}$ does not exist. $\lim_{x\to 0} \frac{x}{x} = 1$

   T F $\frac{d}{dx}(\frac{1}{x}) = -1$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. Using the *definition*, find the derivative of $f(x) = \sqrt{4x - \frac{3}{2}}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{4(x+h)-3/2} - \sqrt{4x-3/2}}{h} \cdot \frac{\left(\sqrt{4(x+h)-3/2} + \sqrt{4x-3/2}\right)}{''}$$

$$= \lim_{h\to 0} \frac{\left(4(x+h)-3/2\right) - \left(4x-3/2\right)}{h\left(\sqrt{4(x+h)-3/2} + \sqrt{4x-3/2}\right)} = \lim_{h\to 0} \frac{\cancel{4x}+4h-\cancel{3/2}-\cancel{4x}+\cancel{3/2}}{h\left(\sqrt{4(x+h)-3/2}+\sqrt{4x-3/2}\right)}$$

$$= \lim_{h\to 0} \frac{4\cancel{h}}{\cancel{h}\left(\sqrt{4(x+h)-3/2}+\sqrt{4x-3/2}\right)} = \lim_{h\to 0} \frac{4}{\sqrt{4(x+h)-3/2}+\sqrt{4x-3/2}}$$

note $\lim_{h\to 0} \sqrt{4(x+h)-3/2} + \sqrt{4x-3/2} = \sqrt{4x-3/2} + \sqrt{4x-3/2} = 2\sqrt{4x-3/2}$

so by our limit laws

$$f'(x) = \lim_{h\to 0} \frac{4}{\sqrt{4(x+h)-3/2}+\sqrt{4x-3/2}} = \frac{4}{2\sqrt{4x-3/2}} = \frac{2}{\sqrt{4x-3/2}}$$

Check: $\frac{d}{dx}\left[(4x-3/2)^{1/2}\right] = \frac{1}{2}(4x-3/2)^{-1/2}\cdot 4$ ✓

3. Given that $f(x)$ is a differentiable function and that a and k are constants, complete the following Derivative Rules:

(a) $\frac{d}{dx}(a^k) = 0$

(b) $\frac{d}{dx}(f(x)^k) = k f(x)^{k-1} \cdot f'(x)$

(c) $\frac{d}{dx}(a^{f(x)}) = a^{f(x)} \cdot \ln a \, f'(x)$

$\frac{d}{dx}\left(e^{f(x)\ln a}\right) = e^{f(x)\ln a} \cdot \ln a \, f'(x)$

(d) $\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} f'(x)$

(e) $\frac{d}{dx}(e^{f(x)}) = e^{f(x)} f'(x)$

(f) $\frac{d}{dx}(\log_a(f(x)) = \frac{1}{f(x)\ln a} f'(x)$

4. Prove if f and g are differentiable, then $\frac{d}{dx}(f-g) = \frac{d}{dx}f - \frac{d}{dx}g$. *Hint:* use the *definition* of a derivative.

$$\frac{d}{dx}(f-g) = \lim_{h\to 0} \frac{(f-g)(x+h)-(f-g)(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-g(x+h)-f(x)+g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)-g(x+h)+g(x)}{h}$$ rearranging terms

$$= \lim_{h\to 0} \frac{[f(x+h)-f(x)]-[g(x+h)-g(x)]}{h}$$ factoring -1

$$= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} - \frac{g(x+h)-g(x)}{h}\right)$$ by prop of fract.

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$ by prop of lim

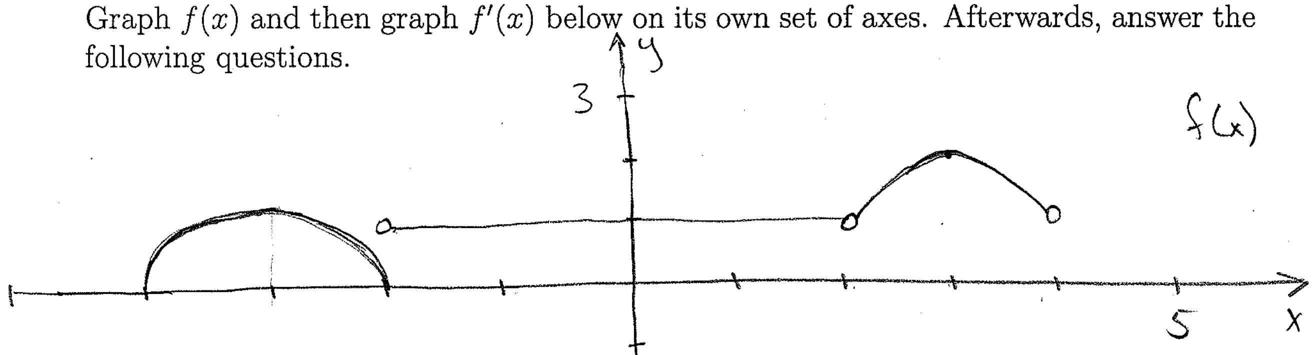
$$= \frac{d}{dx}f - \frac{d}{dx}g$$ by def of der.

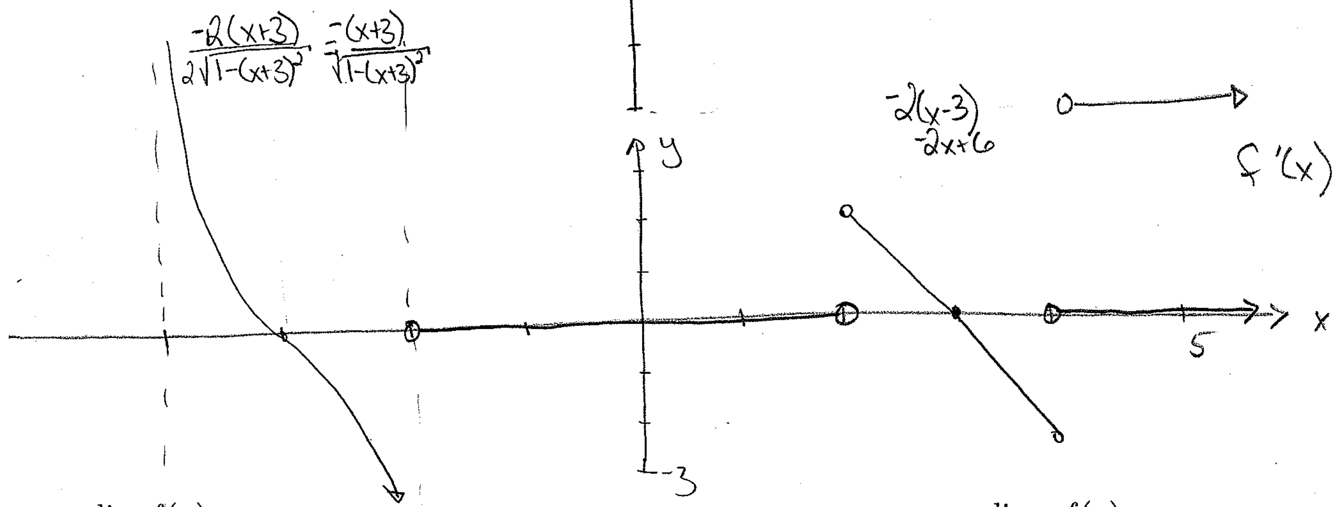
5. Let $f(x) = \begin{cases} \sqrt{1-(x+3)^2} & \text{if } -4 \le x \le -2 \\ 1 & \text{if } -2 < x < 2 \\ -(x-3)^2+2 & \text{if } 2 < x < 4 \\ -3 & \text{if } 4 < x \end{cases}$

upper circle w/ center (-3,0) & radius 1

parabola opening down w/ vertex (3,2)

Graph $f(x)$ and then graph $f'(x)$ below on its own set of axes. Afterwards, answer the following questions.





$\lim_{x \to 4} f(x)$

DNE

$\lim_{x \to 4^+} f(x) = -3 \neq 1 = \lim_{x \to 4^-} f(x)$

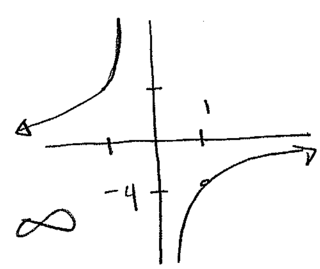
$\lim_{x \to -2^+} f(x) = 1$

$\lim_{x \to -2^-} f'(x) = -\infty$

$\lim_{x \to \infty} f(x) = -3$

6. What is $\lim_{x \to 0} (\frac{-4}{x})$? Explain your answer. DNE

Note the graph of $-4/x$ is



$\therefore \lim_{x \to 0^+} \frac{-4}{x} = -\infty$ but $\lim_{x \to 0^-} \frac{-4}{x} = \infty$

since the limits dont match, $\lim_{x \to 0} \frac{-4}{x}$ DNE.

7. Find $\lim_{x \to 0} x^4 sin(\frac{1}{x})$. Recall that $sin(\frac{1}{x})$ oscillates between -1 and 1 as it gets closer to zero. Explain your reasoning. We'll use the squeeze theorem.

Note $-1 \leq \sin 1/x \leq 1$ for all values of $x \neq 0$
Since $x^4 \geq 0$ the above inequalities are preserved by mult. by x^4, thus we have:
$$-x^4 \leq x^4 \sin 1/x \leq x^4.$$
Since $\lim_{x \to 0} -x^4 = 0 = \lim_{x \to 0} x^4$ by the squeeze theorem $\lim_{x \to 0} x^4 \sin 1/x = 0$.

8. Prove that the function $f(x) = x^3 - 5$ has a *fixed point*. (i.e. show that there exists a point p such that $f(p) = p$)

We need to find a p so that $p = p^3 - 5$
or $0 = p^3 - p - 5$.
Thus it suffices to show $p^3 - p - 5$ has a root.
I'll make use of the I.V.T
Note $0^3 - 0 - 5 = -5 < 0$ and $2^3 - 2 - 5 = 1 > 0$.
Since $p^3 - p - 5$ is cont $\exists$ a value q so that $q^3 - q - 5 = 0 \Rightarrow q^3 - 5 = q$
$\Rightarrow q$ is a fixed point of $f(x)$.

10. Find $\frac{dy}{dx}$ for each of the following:

$y = x \sin \frac{1}{x}$

$$\frac{dy}{dx} = x\left(\cos\frac{1}{x}\right)\frac{-1}{x^2} + \sin{}^1\!/\!_x$$

$$= \frac{-1}{x}\cos\frac{1}{x} + \sin{}^1\!/\!_x$$

$y = x^{x^x}$

$$\ln y = x^x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = x^x \cdot \frac{1}{x} + \ln x \frac{d}{dx}(x^x)$$

let $z = x^x$

$$\ln z = x \ln x$$

$$\frac{1}{z}\frac{dz}{dx} = x \cdot \frac{1}{x} + \ln x$$

$$\frac{dz}{dx} = \frac{d}{dx}(x^x) = x^x(1+\ln x)$$

$$\frac{dy}{dx} = x^{x^x}\left(x^{x-1} + x^x(1+\ln x)\ln x\right)$$

$x^2y^2 = 4 - y\arctan(5x)$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

I'll give you the arc trig identities!

$$x^2 2y\frac{dy}{dx} + 2xy^2 = y\left(\frac{1 \cdot 5}{1+(5x)^2}\right) + \frac{dy}{dx}\arctan(5x)$$

$$\frac{dy}{dx}\left(x^2 2y - \arctan(5x)\right) = \frac{5y}{1+25x^2} - 2xy^2$$

$$\frac{dy}{dx} = \frac{\frac{5y}{1+25x^2} - 2xy^2}{2x^2y - \arctan 5x}$$

$y = \sqrt{x}e^{x^7}(x^6+3)^{10}$

$$\ln y = \tfrac{1}{2}\ln x + x^7 \ln e + 10\ln(x^6+3)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + 7x^6 + \frac{10 \cdot 6x^5}{x^6+3}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2x} + 7x^6 + \frac{60x^5}{x^6+3}\right)$$

12. Suppose that height of a ball from the floor at time t, is described by the equation $H(t) = -t^2 + 7t + 8$.

(a) When is the ball on the floor?

$$0 = -t^2 + 7t + 8$$
$$0 = -(t^2 - 7t - 8)$$
$$0 = -(t-8)(t+1)$$
$$t = 8, -1$$

(b) Find when the ball is at its maximum height.

$v(t) = -2t + 7$ max height when $v(t) = 0$

$$0 = -2t + 7$$
$$3.5 = \frac{7}{2} = t$$

(c) Find the acceleration of the ball at $t = 1$.

$a(t) = -2$ $\therefore -2$

$$\begin{array}{r} 18 \\ \times\ 4 \\ \hline 72 \end{array}$$

$$H(\tfrac{7}{2}) = -\frac{49}{4} + \frac{49}{2} + 8$$
$$= \frac{49}{4} + \frac{32}{4} = \frac{81}{4}$$
$$H(5) = -25 + 35 + 8 = 18$$

(d) Find the total distance traveled from $t = 0$ to $t = 5$.

The ball moves up till $t = 7/2$ & then retraces its position

$\therefore$

Dist covered $t=0$ to $t=7/2$ $\quad |H(0) - H(\tfrac{7}{2})| = |8 - \frac{81}{4}| = |\frac{32-81}{4}| = \frac{49}{4}$

Dist covered $t=7/2$ to $t=5$ $\quad |H(\tfrac{7}{2}) - H(5)| = |\frac{81}{4} - 18| = |\frac{81-72}{4}| = \frac{9}{4}$

Total dist $= |H(0) - H(\tfrac{7}{2})| + |H(\tfrac{7}{2}) - H(5)| = \frac{49}{4} + \frac{9}{4} = \frac{58}{4} = \frac{29}{2} = 14.5$

13. Find the following *limits*

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}} = \sqrt{\lim_{x\to\infty} \frac{x^2+2}{2x^2+1}} = \frac{1}{\sqrt{2}}$$

$$\lim_{x\to\infty} \frac{x^2+2}{2x^2+1} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{4x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

$$e^{\frac{1}{x}\ln x} = x^{1/x}$$

consider $\lim_{x\to\infty} \frac{1}{x}\ln x$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\therefore \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^0 = 1$$

14. Fill out the following and then graph $f(x) = 2\cos x + \sin 2x$

I would have given you the double angle id.

$\sin 2x = 2\sin x \cos x$

$\cos 2x = 1 - 2\sin^2 x$

$= \cos^2 x - \sin^2 x$

- Domain: $\mathbb{R}$
- x-intercepts: $y=0$ $\quad 0 = 2\cos x + 2\sin x\cos x$
  $= 2\cos x\,(1+\sin x)$ $\quad x = \frac{\pi}{2} + 2\pi k,\ \frac{3\pi}{2} + 2\pi k$
- y-intercepts:
  $(0,2)$ $\quad x=0$ $\quad y = 2\cos 0 + \sin 2\cdot 0 = 2 + 0 = 2$
- Symmetry of $f(x)$: periodic w/ period 2π
  (since $\sin 2x$ has period π
- vertical asymptotes:
  Since the domain is $\mathbb{R}$ there are no v.a.
- horizontal asymptotes:
  $\lim_{x\to\infty}(2\cos x + \sin 2x) = \lim_{x\to\infty} 2\cos x(1+\sin x)$ $\quad$ trig functions are periodic so there are no h.a.
- extrema (both x and y coordinates)
  $f'(x) = 2\cos x(\cos x) - (1+\sin x)2\sin x = 2(\cos^2 x - \sin x - \sin^2 x)$
- intervals $f(x)$ is increasing:
  $[0, \pi/6) \cup (5\pi/6, 2\pi]$
- pts. of inflection (both x and y coordinates)
  $f''(x) = 2[(-2\sin x + 1)(\cos x) + (\sin x + 1)(-2\cos x)]$ ★
- intervals $f(x)$ is concave up
- Graph $f(x)$

I'll give more room on exam.

$= 2(1 - 2\sin^2 x - \sin x)$

$= 2(-2\sin x + 1)(\sin x + 1)$

if $f'(x) = 0$ then

$\sin x = \frac{1}{2}$ or $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad x = \frac{3\pi}{2}$

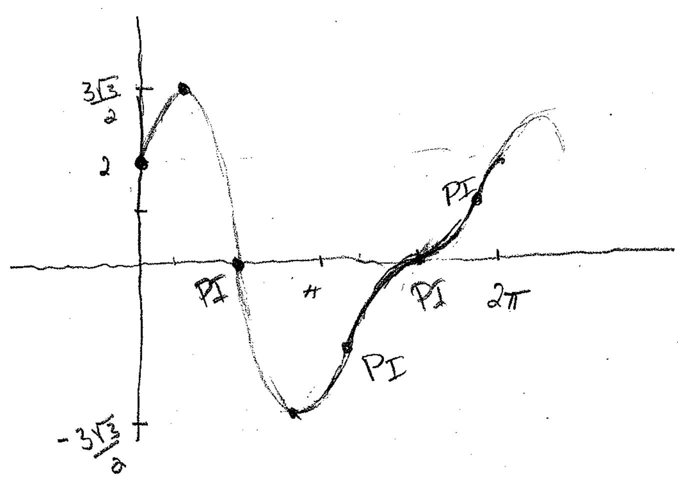
$+ \quad - \quad + \quad +$

$0 \quad \frac{\pi}{6} \quad \frac{5\pi}{6} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

max $\quad$ min $\quad$ not extreme

max $\left(\frac{\pi}{6}, 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{3}{2}\right)$

min $\left(\frac{5\pi}{6}, -2\frac{\sqrt{3}}{2}\cdot\frac{3}{2}\right)$



★ $f''(x) = 2\cos x(-2\sin x + 1 - 2\sin x - 2)$

$= 2\cos x(-4\sin x - 1)$

if $f''(x) = 0$ then

$\cos x = 0$ $\quad \sin x = -\frac{1}{4}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ hard to find between $\pi, 7\pi/6$; $11\pi/6$ & 2π

$- \quad + \quad - \quad + \quad -$

$0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

~~18~~. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3m deep.

Recall the volume of a cone is $\frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is the height of the cone.

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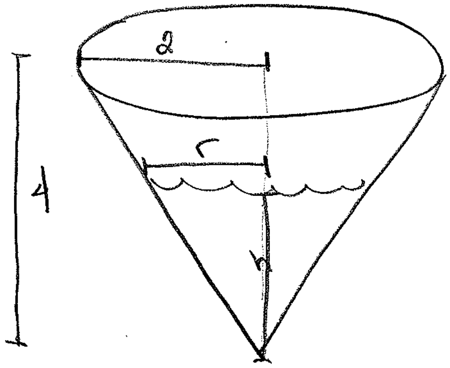
Let V be the volume of the water

$$\frac{dV}{dt} = 2 \,\text{m}^3/\text{min}$$

Let h be the height of the water.

we want $\frac{dh}{dt}$



Need to find a relationship between $\frac{dV}{dt}$ & $\frac{dh}{dt}$.

Know $V = \frac{1}{3}\pi r^2 \cdot h$

If we can write V only as a function of h (instead of h & r) we could take derivatives.

By similar triangles $\frac{r}{h} = \frac{2}{4} \Rightarrow r = \frac{1}{2}h$

Thus $V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{3}\frac{1}{4}h^3$

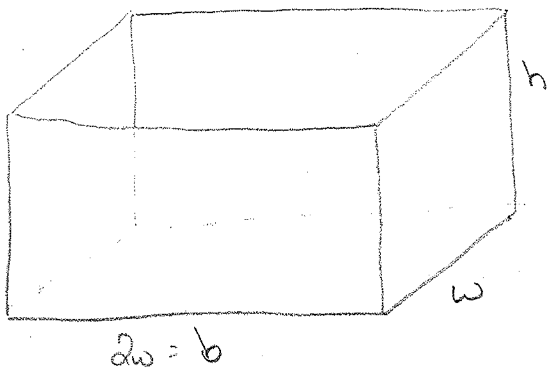
$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{12}\cdot 3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

We're looking for $\frac{dh}{dt}$ when $h=3$ so

$$2 = \frac{\pi}{4}(3)^2\frac{dh}{dt}$$

$$\Rightarrow \frac{8}{9\pi} = \frac{dh}{dt}$$

14. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.



$b \cdot w \cdot h = 10$

$b = 2w$

Cost of base $= 10 \cdot b \cdot w$
$= 10 \cdot 2w \cdot w$
$= 20w^2$

Cost of sides $= 6\,[wh + bh + wh + bh]$
$= 6\,[2wh + 2bh]$
$= 6\,[2wh + 2(2w)h]$
$= 6\,[2wh + 4wh]$
$= 6\,[6wh]$
$= 36wh$

Total Cost $= 20w^2 + 36wh$

we'd like to minimize this

recall $10 = b \cdot w \cdot h = 2w \cdot w \cdot h$

$\Rightarrow 10 = 2w^2 h$

$\Rightarrow \dfrac{10}{2w^2} = h$

$\Rightarrow 5/w^2 = h$

$C(w) =$ Total Cost $= 20w^2 + 36w \cdot \dfrac{5}{w^2} = 20w^2 + \dfrac{180}{w}$

$C'(w) = 40w - \dfrac{180}{w^2}$

critical points

$0 = 40w - 180/w^2$

$\dfrac{180}{w^2} = 40w$

$\dfrac{9}{2} = \dfrac{180}{40} = w^3$

$\Rightarrow \sqrt[3]{\dfrac{9}{2}} = w$

$C'(w)$: $40(\frac{1}{10}) - \frac{180}{\frac{1}{100}}$ $-$; $40(10) - \frac{180}{100}$ $+$

$0 \searrow \sqrt[3]{\frac{9}{2}} \nearrow$

min

Minimal Cost would be

$C(\sqrt[3]{\tfrac{9}{2}}) = 20\left(\sqrt[3]{\tfrac{9}{2}}\right)^2 - \dfrac{180}{\sqrt[3]{\tfrac{9}{2}}}$