MIDTERM 2

Math 251

Spring 2008

Name: KEY

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{x^2+1}$

Show your work for the following problems. You need only simplify if the question explicitly asks for it.

1. [4] Use any results covered in class to find the following:

$$\lim_{x\to 0} \frac{\sin x}{x}$$

(+1) 1

notation (+1)

$$\lim_{x\to 0} \frac{3\sin(4x)}{2\sin(3x)}$$

(+1) mult by $\frac{4x}{4x}$ or $\frac{\left(\frac{1}{3x}\right)}{\left(\frac{1}{3x}\right)}$

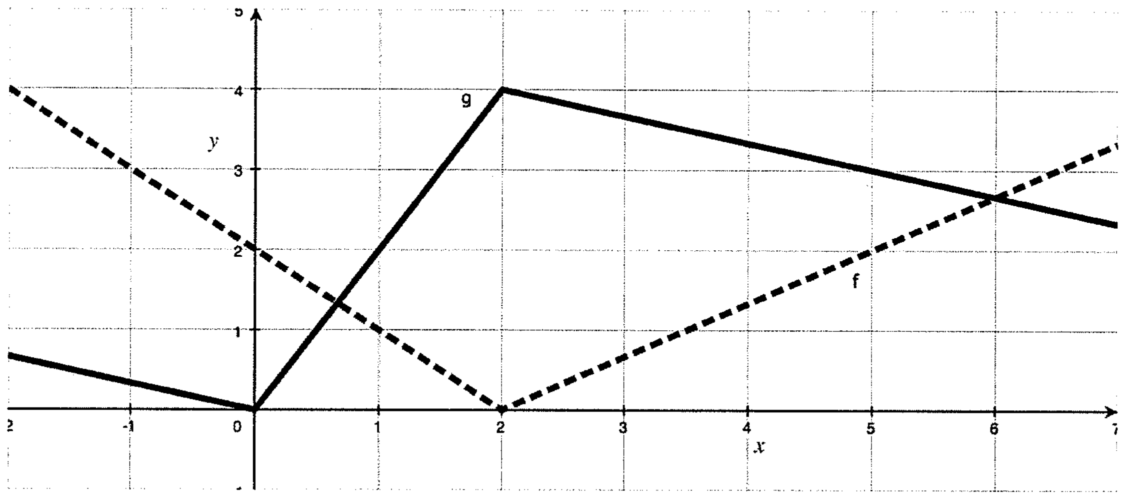
$$= \lim_{x\to 0} \frac{3}{2}\,\frac{\sin 4x}{4x}\cdot 4x \cdot \frac{\left(\frac{1}{3x}\right)}{\left(\frac{\sin 3x}{3x}\right)}$$

$$= \cancel{\lim_{x\to 0}\frac{\sin 4x}{4x}}^{\,1}\ \lim_{x\to 0} \frac{12x}{2}\cdot\frac{1}{3x}\cdot \cancel{\lim_{x\to 0}\frac{1}{\left(\frac{\sin 3x}{3x}\right)}}^{\,1}$$

$$= \lim_{x\to 0}\frac{12x}{6x} = \lim_{x\to 0} 2 = 2$$

properties used right inc. trig (+1)

2. [12] Let the graph of f and g be those shown below.



Define

$h(x) = 5f(x) - 4g(x)$

$j(x) = f(g(x))$

$u(x) = f(x)g(x)$

$v(x) = f(x)/g(x)$

Find the following:

$h'(1)$

dist +1

$h'(x) = 5f'(x) - 4g'(x)$

$h'(1) = 5f'(1) - 4g'(1)$

$= 5(-1) - 4(2)$ +1 +1

$= -5 - 8 = -13$

$j'(5)$

chain +1

$j'(x) = f'(g(x))\cdot g'(x)$

$j'(5) = f'(g(5))\cdot g'(5)$ +1

$= f'(3) \cdot -\frac{1}{3}$

$= \frac{2}{3} \cdot -\frac{1}{3} = -\frac{2}{9}$ +1

$u'(1)$

product +1

$u'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$u'(1) = f'(1)\cdot g(1) + f(1)\cdot g'(1)$

$= (-1)\cdot 2 + 1\cdot 2$ +1 +1

$= -2 + 2 = 0$

$v'(5)$

quotient +1

$v'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{(g(5))^2}$

$= \dfrac{3 \cdot \frac{2}{3} - 2\left(-\frac{1}{3}\right)}{3^2}$ +1 +1

$= \dfrac{2 + \frac{2}{3}}{9} = \frac{8}{3} \cdot \frac{1}{9} = \frac{8}{27}$

3. [15] Find the derivatives of the following:

$g(x) = 2^{\log_2(\pi x)}$ $= \pi x$

$g'(x) = \pi$ (+2)

power rule

notation (+2)

$h(x) = \ln \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

prop of ln (+1)

$= \ln x^{3/4} + \ln(x^2+1)^{1/2} - \ln(3x+2)^5$

$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$

der (+1)

$h'(x) = \frac{3}{4}\frac{d}{dx}(\ln x) + \frac{1}{2}\frac{d}{dx}(\ln(x^2+1)) - 5\frac{d}{dx}(\ln(3x+2))$

$f(x) = \ln x \quad f'(x) = \frac{1}{x}$

$g(x) = x^2+1 \quad g'(x) = 2x$

$= \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\cdot\frac{1}{3x+2}\cdot 3$

$= \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$

or

is do with quotient (+1)

product inside for two (+1)

correct (+1)

$m(x) = \frac{1}{x}\arcsin(x)$

product (+1)

$m'(x) = \frac{1}{x}\frac{d}{dx}(\arcsin x) + \frac{d}{dx}\left(\frac{1}{x}\right)\arcsin x$

$= \frac{1}{x}\cdot\frac{1}{\sqrt{1-x^2}} + \underbrace{\frac{-1}{x^2}}_{(+1)}\arcsin x$

correct (+1)

– or –

let $m(x) = y$ then $yx = \arcsin x$ (+1)

$\Leftrightarrow \sin(xy) = x$

$\frac{d}{dx}(\sin(xy)) = \frac{d}{dx}(x)$

$(\cos xy)(x\frac{dy}{dx} + y) = 1$ (+1)

$\frac{dy}{dx} = \left(\frac{1}{\cos xy} - y\right)\frac{1}{x}$

solved (+1)

$y = \arcsin x \Leftrightarrow \sin y = x$

$\left(\frac{d}{dx}\right.$

$\frac{dy}{dx}\cos y = 1$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$n(x) = (\cos x)^x$

know to use logs (+1)

let $n(x) = y$ then

$y = (\cos x)^x$

prop (+1)

$\Rightarrow \ln y = \ln(\cos x)^x = x\ln(\cos x)$

$\frac{d}{dx}$ (

$\overbrace{\frac{1}{y}\frac{dy}{dx}}^{(+1)} = x\frac{d}{dx}(\ln(\cos x)) - \frac{d}{dx}(x)\ln(\cos x)$

$f(x) = \ln x \quad f'(x) = \frac{1}{x}$

$g(x) = \cos x \quad g'(x) = -\sin x$

$\frac{dy}{dx} = y\left(\overbrace{x\cdot\frac{1}{\cos x}\cdot -\sin x}^{(+1)} + \ln\cos x\right)$

$= y(-x\tan x + \ln\cos x)$

solved for $\frac{dy}{dx}$ (+1)

4. [6] *Prove* the following using only the definition of a derivative and limit properties. Clearly explain each of your steps.

If c is a real number and f is a differentiable function for all x, then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

$\frac{d}{dx}(cf(x)) = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h}$ by def of $\frac{d}{dx}$ (+2) – 1 missing lim

$= \lim_{h \to 0} \frac{c[f(x+h) - f(x)]}{h}$ by alg (+1)

$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ by prop of limits. (+1)

$= c \frac{d}{dx} f(x)$ by def of $\frac{d}{dx}$

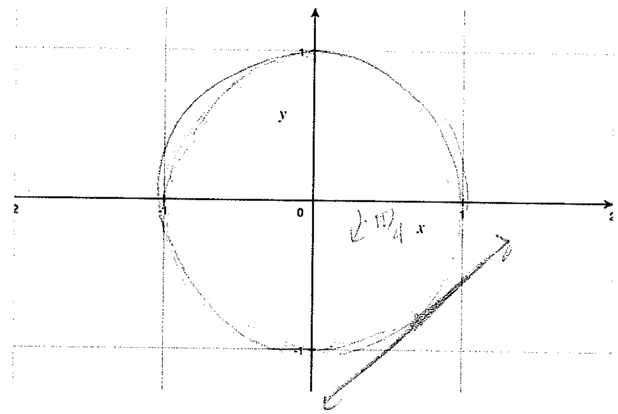
notation (+1)

sense (+1)

if use product rule give up to 2 points

5. Consider the relation described by $x^2 + y^2 = 1$.

(a) [2] Draw the collection of ordered pairs (x, y) that satisfy the relation $x^2 + y^2 = 1$.



unit circle

(b) [1] On the graph above, draw the line tangent to the graph of $x^2 + y^2 = 1$ at $(\cos\frac{-\pi}{4}, \sin\frac{-\pi}{4})$.

(c) [7] Find the equation of the tangent line you just drew.

looking for line (+1)

notation (+1)

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$ Knew to take der (+1)

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$ found it (+1)

$m = \frac{dy}{dx}\Big|_{(\cos\frac{-\pi}{4}, \sin\frac{-\pi}{4})} = \frac{-\cos\frac{\pi}{4}}{\sin\frac{-\pi}{4}} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$ (+1)

$y = mx + b$

$-\frac{1}{\sqrt{2}} = 1\frac{1}{\sqrt{2}} + b$ plug in for line (+1)

$\Rightarrow \frac{-2}{\sqrt{2}} = b$

So $y = 1x - \frac{2}{\sqrt{2}}$

evaluated the angles correctly (+1)

(d) [3] Let θ be the measure of an angle between 0 and 2π starting from the positive x-axis. Give the coordinates where the terminal side intersects the graph in terms of θ.

$(\cos\theta, \sin\theta)$ notation (+1)

(+1) (+1)

(e) (Extra Credit!) Find the equation of the line that is tangent to the graph of $x^2 + y^2 = 1$ at the coordinates you just wrote down for (d).

$\frac{dy}{dx} = \frac{-x}{y}$

So $\frac{dy}{dx}\Big|_{(\cos\theta, \sin\theta)} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$ found der +1

$y = mx + b$

$\sin\theta = (-\cot\theta)(\cos\theta) + b$

$\Rightarrow b = \sin\theta + (\cot\theta)\cos\theta$

$\therefore y = (-\cot\theta)x + \sin\theta + \cot\theta\cos\theta$

$= (-\cot\theta)x + \csc\theta$ +1