KEY

Implicit Differentiation

§ 3.5

1. Given that $\frac{1}{x} + \frac{1}{y} = 1$,

   (a) find y' by implicit differentiation,

   $$\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{d}{dx}(1)$$
   $$\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{d}{dx}\left(\frac{1}{y}\right) = 0$$
   $$\underset{\text{by } *}{-\frac{1}{x^2}} + \underset{\text{by } \dagger}{-\frac{1}{y^2}\frac{dy}{dx}} = 0$$
   $$-\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x^2}$$
   $$\boxed{\frac{dy}{dx} = -\frac{y^2}{x^2}}$$

   $$* \ \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2}$$
   $$\dagger \ \frac{d}{dx}\left(\frac{1}{y}\right)$$
   $$f(x) = 1/x \qquad f'(x) = -x^{-2}$$
   $$g(x) = y \qquad g'(x) = dy/dx$$
   $$f(g(x)) = f(y) = 1/y \ \checkmark$$
   $$\frac{d}{dx}\left(\frac{1}{y}\right) = f'(g(x))g'(x) = \frac{-1}{y^2}\frac{dy}{dx}$$

   (b) solve for y in the given equation in terms of x,

   $$\frac{1}{x} + \frac{1}{y} = 1$$
   $$\frac{1}{y} = 1 - \frac{1}{x} = \frac{x-1}{x}$$
   $$1 = \left(\frac{x-1}{x}\right)\cdot y$$
   $$\boxed{\frac{x}{x-1} = y}$$

   (c) differentiate the result of (b) get y', and

   $$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{x}{x-1}\right) = \frac{(x-1)\frac{d}{dx}(x) - x\frac{d}{dx}(x-1)}{(x-1)^2}$$
   $$= \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

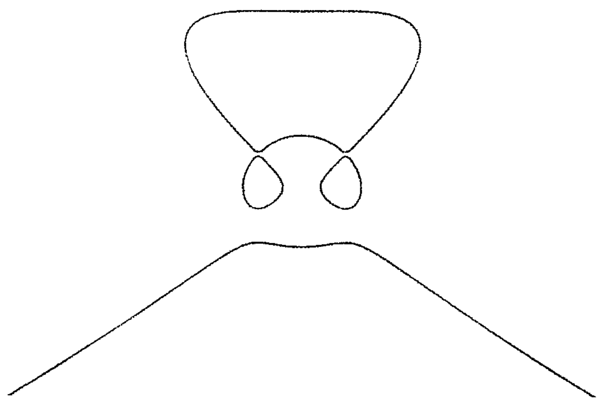
   (d) check that your answer in part (a) agrees with part (c) by substituting the expression for y from part (b) into the answer in part (a).

   part a: $$\frac{dy}{dx} = -\frac{y^2}{x^2} = \underbrace{-\frac{\left(\frac{x}{x-1}\right)^2}{x^2}}_{\text{sub part b}} = -\frac{x^2}{(x-1)^2} \div x^2 = \frac{-\cancel{x^2}}{(x-1)^2}\cdot\frac{1}{\cancel{x^2}}$$
   $$= \frac{-1}{(x-1)^2} = \text{part c}$$

2. It is mentioned in exercise 38 that the graph of the equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2,$$

as seen below without axes, looks like a bouncing wagon.



Find y' using implicit differentiation.

$\frac{d}{dx}(2y^3 + y^2 - y^5) = \frac{d}{dx}(x^4 - 2x^3 + x^2)$

$2\frac{d}{dx}(y^3) + \frac{d}{dx}(y^2) - \frac{d}{dx}(y^5) = \frac{d}{dx}(x^4) - 2\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$

$f(x) = x^3 \quad f'(x) = 3x^2 \qquad f(x) = x^2 \quad f'(x) = 2x$

$g(x) = y \quad g'(x) = \frac{dy}{dx} \qquad g(x) = y \quad g'(x) = dy/dx$

$2f'(g(x)) \cdot g'(x) + f'(g(x))g'(x) - \frac{d}{dx}(y^5) = 4x^3 - 2 \cdot 3 \cdot x^2 + 2x$

$2 \cdot 3y^2 \frac{dy}{dx} + 2y\frac{dy}{dx} - 5y^4\frac{dy}{dx} = 4x^3 - 6x^2 + 2x$

$\frac{dy}{dx}(6y^2 + 2y - 5y^4) = 4x^3 - 6x^2 + 2x$

$\frac{dy}{dx} = \frac{4x^3 - 6x^2 + 2x}{6y^2 + 2y - 5y^4}$

1pt completion
1pt first pg
1pt second pg